Complex Dynamics with Illustrations using Mathematica

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Abstract

The field of complex dynamics has undergone rapid development in the past few decades resulted from the ever-surging operation speed and capacity of modern computers. Abstract concepts can now be visualized through computer graphics. The aim of this paper is two-fold; firstly, to make a survey over the complex dynamics of rational functions; secondly, to illustrate the abstract concepts through examples and their computer graphics.

The study of complex dynamics begins with the study of local behaviour of rational functions under iteration, especially for the iteration near a fixed point (or a periodic point). In chapter 4, we discuss the orbits of any chosen point, z_0 near a fixed point, it was noted that for different fixed points, the behaviour of the orbits may be different and be differentiated by the different classification of the fixed points according to the multiplier associated with it.

In order to study the topic in a more global point of view, basic knowledge in complex analysis and general topology is needed. In chapter 1 and 2, we give an account on the results needed in complex analysis and topology, they serve as a prerequisite for deriving the Riemann-Hurwritz relation and the Montel's theorem which play an important role in the study of complex dynamics and will be discussed in chapter 3.

In chapter 5 and 6, Fatou set and Julia set were introduced under the iteration of rational functions, also, we will have a further investigation into the properties of the Julia set and Fatou set. Besides adopting results from complex analysis, computer graphics were also used to explore the complex dynamics. Examples had been constructed and graphics

were produced by routine written in Mathematica to serve the purpose of illustrating and displaying the abstract concepts.

In the final chapter, the crucial role of critical points and their orbits taken in the complex dynamics were discussed, the dynamics by iterating a polynomial can serve as an example illustrating the fact and in the final section of the chapter, we had a discussion on the dynamics of iterating quadratic polynomials since we had a thorough understanding of the topics due to the introduction and the study of the Mandelbrot set.

Contents

Chapter 1 Fundamentals of Complex Analysis

1.1 The extended complex plane

Any complex number with finite modulus can be represented as a point on the complex plane (which may be called the Argand plane). As in most of our discussion in the topic of complex dynamics, an abstract point, representing infinity which is denoted by ∞, is so important that it cannot be omitted. When we study the iteration of function, we often encounters the algebraic operation involving ∞, its manipulation with finite number, a, is defined by setting the following rules:

(1) $a + \infty = \infty + a = \infty$

(2) $a \cdot \infty = \infty$ where $a \neq 0$.

(3) $\dfrac{a}{0} = \infty$ for $a \neq 0$

(4) $\dfrac{b}{\infty} = 0$ for $b \neq \infty$

However, it is impossible to define $\infty + \infty$ and $0 \cdot \infty$, as it would violate the law of arithmetic. For evaluating the function at ∞, or more precisely, we are finding the limit of the function when the independent variable tend to ∞. Suppose a function $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$ has a finite limit l when z tends to ∞, i.e. $\lim_{z \to \infty} f(z) = l$ then for all $\varepsilon > 0$, there is a real number r such that $|f(z) - l| < \varepsilon$ when $|z| > r$. On the other hand, if the value of a function f tends to infinite when the independent variable tends to a complex number, a, with finite modulus, i.e. $\lim_{z \to a} f(z) = \infty$, then for all $\varepsilon > 0$, there is a

$\delta > 0$ such that $|f(z)| > \varepsilon$ whenever $|z - a| < \delta$.

The importance of introducing ∞ in the iteration of functions is that it maintain the onto condition for a rational map so that pullback of a point or a set can always be found, that is, backward iteration can be performed for as many times as we wish. Let us consider an example, given $f(z) = \dfrac{az+b}{cz+d}$, then ∞ is the inverse image of $\dfrac{a}{c}$.

As we know that there is no room for the point ∞ in the complex plane, so we adjoin the point, ∞ , to the complex plane **C** to form an extended plane $\mathbf{C}_\infty$. Thus $\mathbf{C}_\infty$ is simply the union of the complex plane **C** and the point ∞.

1.2 Stereographic projection

In order to visualise the effect of iteration of functions on the neighbourhood of ∞, it is desirable to represent each point on the extended plane on a geometric model. Let $\mathbf{S}^2$ be the sphere in $\mathbf{R}^3$ with unit radius and the centre at the origin, every point (x_1, x_2, x_3) on $\mathbf{S}^2$ satifies the equation $x_1{}^2 + x_2{}^2 + x_3{}^2 = 1$. Define $\psi : \mathbf{S}^2 \to \mathbf{C}_\infty$.

$\psi(x_1, x_2, x_3) = \dfrac{x_1 + ix_2}{1 - x_3}$ where $(x_1, x_2, x_3) \neq (0, 0, 1)$, otherwise $\psi(x_1, x_2, x_3) = \infty$.

It's not difficult to show that

$\psi^{-1}(z) = \left(\dfrac{x}{1+|z|^2}, \dfrac{y}{1+|z|^2}, \dfrac{|z|^2}{1+|z|^2}\right)$ where $z = x + iy$ with $|z|$ is finite, otherwise

$\psi^{-1}(z) = (0, 0, 1)$.

It's easy to see that ψ and ψ^{-1} are one-one and onto, also, they are continuous, therefore ψ is a homeomorphism. Thus, we may regard the sphere as a representation of the extended complex plane. Moreover, ψ takes on a simple geometrical meaning, if **C** is identified with the horizontal plane $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = 0\}$, then for any point $x + iy$ on **C**, we have a relation that $x : y : -1 = x_1 : x_2 : x_3 - 1$ which means that $(x, y, 0)$, (x_1, x_2, x_3) and $(0, 0, 1)$ are collinear. Hence the correspondence is a central projection from the centre $(0, 0, 1)$ as shown in figure 1.1 and ψ is called a *stereographic projection.*

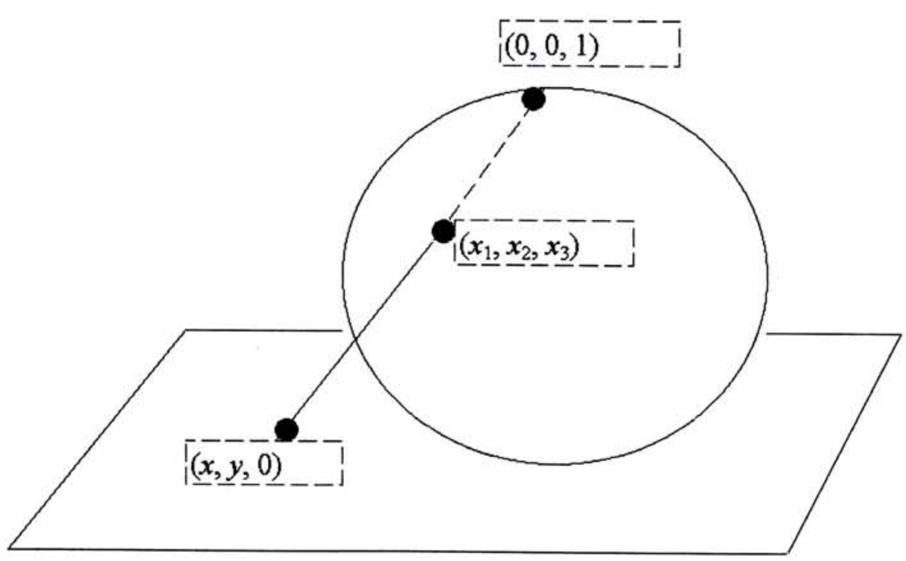


Figure 1.1

1.3 Analytic Functions

Given a map $f : D \to \mathbf{C}$ from the planar domain D of **C** to **C**. It is *analytic* on a domain D if and only if it possesses a *derivative* for each point on D. The definition for analyticity can easily be extended to the whole complex plane **C**. If $D \subseteq \mathbf{C}_\infty$, it remains to define the analyticity at ∞. Let J be the map $J(z) = \dfrac{1}{z}$, then f is said to be defined in some neighbourhood of ∞ if it is defined on some set $V = \{|z| > r\} \cup \{\infty\}$ where r is a real number. It is important to note that $J(V) = \{|z| < \frac{1}{r}\}$. f is analytic at ∞ if $f \circ J$ is analytic in some neighbourhood of the origin. Thus, for a function f to be analytic on the extended complex plane, it should be analytic on both the complex plane and ∞ .

A map $f : D \to \mathbf{C}_\infty$ is analytic on the plane domian D if each point of D has a neighbourhood on which either f or $\dfrac{1}{f}$ is analytic. For the poles of f which are the points satisfy $f(\omega) = \infty$, we see that there is a neighbourhood of such points that the map $z \mapsto \dfrac{1}{f(z)}$ is analytic with value zero at ω , that is, by our definition analytic at such points. Also f is analytic at ∞ if the map $f \circ J$ is analytic near the origin.

As we know from standard text of Complex Analysis, a function f is analytic if and only if it can be expressed locally as a power series. That is, for $z_0 \in D$, there is an $r > 0$ (or a neighbourhood) and complex number a_0, a_1 , a_2 , ..., that

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots , \quad \text{for all } z \text{ with } |z - z_0| < r$$

If a_υ is the first non zero coefficient, then we say that υ is the *valency* of f at z_0.

Given a map $f : D \to \mathbf{C}_\infty$ from the planar domain D to $\mathbf{C}_\infty$, which is analytic according to the above definition, then in usual text of Complex Analysis, it is called *meromorphic*, and there exists z_1 , z_2 , ..., $z_k \in D$ and polynomials P_1 , P_2 , ..., P_k and a function g analytic in D such that

$$f(z) = P_1(\frac{1}{z-z_1}) + \cdots + P_k(\frac{1}{z-z_k}) + g(z) \qquad \text{for all } z \in D.$$

Here z_1 , z_2 , ..., z_k are called the *poles* of f. It is easily seen from the definition that f is analytic in $D\backslash\{z_1, z_2, ..., z_k\}$.

1.4 Rational Functions

A rational function is a function of the form $R(z) = \dfrac{P(z)}{Q(z)}$ where P and Q are polynomials in z, not both being zero polynomial and they are coprime. If P is a zero polynomial then R is a constant function, zero, while, if Q is a zero polynomial then R is a constant function ∞ . As P and Q are coprime, therefore they are uniquely determined up to a multiplication by non-zero constant. The degree of R ($deg(R)$) is defined by $deg(R) = \max\{deg(P) , deg(Q)\}$. It is easy to see that R has precisely $deg(R)$ zeros and $deg(R)$ poles in $\mathbf{C}_\infty$ and Also $deg(R)$ is the number of preimages of any value counting multiplicities.

It is also important to note that for any polynomial P. P is analytic throughout $\mathbf{C}_\infty$. A rational function is a quotient of two polynomials, therefore any rational functions are analytic throughout $\mathbf{C}_\infty$ and that if f is meromorphic in $\mathbf{C}_\infty$, then f is a rational function. To see the above fact, we have first to show that f has only a finite number of poles in $\mathbf{C}_\infty$. Indeed as f is meromorphic in $\mathbf{C}_\infty$, therefore it is meromorphic in $\{z \in \mathbf{C}: |z| > r\} \cup \{\infty\}$ for some $r \in \mathbf{R}$, $f \circ J$ is meromorphic in $C(0 , r^{-1})$. we deduced that for some positive δ, $f \circ J$ has no poles in $C^*(0 , \delta)$ and this means that f itself has no poles in $\{z : |z| > \delta^{-1}\}$.

As the poles of f in **C** are isolated, f can only have a finite number of poles, say z_1 , ..., z_s in **C** and possibly a pole at ∞ . We may now write

$$f(z) = \sum_{j=1}^{s} P_j\left(\frac{1}{z - z_j}\right) + g(z)$$

where P_j are polynomials and g is analytic in **C**. This shows that $g \circ J$ differ from $f \circ J$ by a rational function and since $f \circ J$ is meromorphic in C(0 , r^{-1}), and so is $g \circ J$. As g is analytic in **C**, we may write

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

therefore in $C^*(0\ ,\ r^{-1})$; $g \circ J(z) = g(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n}$.

As $g \circ J$ is meromorphic in C(0 , r^{-1}), therefore there is a real k such that for $n > k$, $a_n = 0$; thus g is a polynomial and therefore f is rational.

1.5 Mobius Transformations

The non constant rational maps of degree one, that is

$$f(z) = \frac{az + b}{cz + d} \qquad \text{with } ad - bc \neq 0$$

are called *Mobius Transformations*. It is known that Mobius Transformation is the only one-one meromorphic map from $\mathbf{C}_\infty$ onto $\mathbf{C}_\infty$ and the inverse map is given by

$$f^{-1}(z) = \frac{dz - b}{-cz + a}$$

As the inverse and identity exists for the set of Mobius Transformations, therefore they can be used to introduce *conjugacy*. Two rational maps R_1 and R_2 are conjugate pair if and only if there exists a Mobius Transformation such that $R_2 = fR_1f^{-1}$

Conjugacy is important for its invariant properties for functions under conjugation. The most imprtant of all is that: if R_1 and R_2 are a conjugate pair, then $\deg(R_1) = \deg(R_2)$. Moreover, if $R_2 = fR_1f^{-1}$, then $R_1 = f^{-1}R_2f$ which means that we can transfer a problem concerning R_1 to a problem concerning its conjugate, R_2 , from which the problem may make simpler, and then attempt to solve the problem in terms of R_2, consequently the job can be accomplished by rewriting the solution in terms of R_1. Also, conjugacy respect fixed point, that is, if $fR_1f^{-1} = R_2$, then R_2 fixes $f(z)$ if and only if R_1 fixes z. The above properties are important , since from now on, when we are doing iteration on rational maps, we shall not bother to distinguish between rational functions. Besides conjugacy, Mobius Transformations are important for its preservation of chordal distance on the sphere. Moreover, it should be noted that for local conjugacy, f must be a bianalytic homeomorphism (i.e. f and f^{-1} are both analytic) and need not be a Mobius transformation.

Before we end this chapter, we give an example of conjugation of quadratic polynomial. Given an arbitrary quadratic plynomial, $R_1(z) = az^2 + bz + c$, By chosing a suitable Mobius Transformation: $f(z) = az + \frac{b}{2}$, the given polynomial is conjugate to another quadratic polynomial, $R_2(z) = z^2 + k$. From the above discussion, in order to study the dynamics of iteration of quadratic polynomials, it suffices to consider only those of the form $R_2(z) = z^2 + k$. Due to the simplicity of producing computer graphics therefore most form $R_2(z) = z^2 + k$. Due to the simplicity of producing computer graphics therefore most of the examples given in this paper are quadratic polynomials. The dynaimcs of the quadratic polynomial depends largely on the values of k and will be discussed in the last section of this paper.

Chapter 2 The Topology of the Extended Plane

As we know that by direct application of properties concerning the topology of the complex sphere, we can get some of the results on the structure of the Fatou set and Julia set which does not require anything from the theory of iteration. Therefore it seems better to take a look at the topology of the extended complex plane, the complex sphere, their relations and also the relevant topological results.

2.1 The Topology of $\mathbf{S}^2$ and $\mathbf{C}_\infty$

Before the quoting of results from topology, we shall take a glance on the topology of the sphere, $\mathbf{S}^2$ and the extended complex plane, $\mathbf{C}_\infty$. Let ω and z be two points on the complex sphere, the distance between the points, denoted by $d(\omega, z)$ can be defined as the length of chord joining the two points (which may be called the chordal distance on $\mathbf{S}^2$). Thus, for each point ω on $\mathbf{S}^2$, we can define a neighbourhood of ω by using the chordal distance, that is $\{x \in \mathbf{S}^2 : d(x, \omega) < \delta\}$ is a neighbourhood for ω, where $\delta \in \mathbf{R}$.

On the extended complex plane, it is easy to see that for a finite complex number z_1, a neighbourhood of z_1 can be defined as $\{z \in \mathbf{C} : |z - z_1| < d\}$ where d is real . For the point ∞ , we define its neighbourhood by $\{ z \in \mathbf{C} : |z| > r\} \cup \{ \infty \}$. As we can define neighbourhood for each point on $\mathbf{S}^2$ and the extended complex plane, then we can define a topology induced by the respective metric.

From our previous discussion, we knew that the stereographic projection is a homeomorphism, therefore $\mathbf{S}^2$ and $\mathbf{C}_\infty$ are homeomorphic, which means that it is not only a one-one function from $\mathbf{S}^2$ to $\mathbf{C}_\infty$ but also a natural one-one correspondence between open sets of $\mathbf{S}^2$ and $\mathbf{C}_\infty$, that is, $\mathbf{S}^2$ and $\mathbf{C}_\infty$ can be viewed as equivalent topologically.

The following results are important in the study of the properties of Fatou set and Julia set and will be stated without proof:

(1) The closure of a connected set is connected.

(2) A compact set K on the sphere is disconnected if and only if there exists a Jordan curve γ which separates K, that is K is disjoint from γ and meet both components of the complement of γ.

(3) A domain is simply connected if and only if its complement is connected.

(4) A domain D is simply connected if and only if its boundary is connected.

(5) Let D be an open subset of the complex sphere, then $\mathbf{C}_\infty \backslash D$ is connected if and only if each component of D is simply connected.

It can be shown that the last three results mentioned above are equivalent.

2.2 Smooth Map and Manifolds

In order to define the degree of a rational function in a topological point of view, we shall introduce the *regular point* of a *smooth manifold*. Let $U \subseteq \mathbf{R}^k$ and $V \subseteq \mathbf{R}^l$ be open sets where $\mathbf{R}^k$ and $\mathbf{R}^l$ are Euclidean space of dimension k and l respectively. A mapping f from U to V is called a *smooth map* if all of the partial derivatives $\dfrac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}$ exist and are continuous.

An n-dimensional topological manifold M^n is a Hausdorff topological space with a countable basis for the topology which is locally homeomorphic to $\mathbf{R}^n$. The last condition means that for each point $p \in \mathbf{R}^n$, there is an open neighbourhood U of p and a homeomorphism $h : U \to U'$ onto an open set $U' \in \mathbf{R}^n$, we illustrate this fact by considering $\mathbf{C}_\infty$ which can be shown as a 2-dimensional topological manifold: Let $U_1 = \mathbf{C}$ and $U_2 = \mathbf{C}_\infty \backslash \{0\}$, then $U_1 \cap U_2 = \mathbf{C}\backslash\{0\}$, define $\varphi_1 : \mathbf{C} \to \mathbf{R}^2$ which is the identiy map and $\varphi_2 : \mathbf{C}_\infty\backslash\{0\} \to \mathbf{R}^2$ such that $z \mapsto \dfrac{1}{z}$. It is clear that φ_1 and φ_2 are homeomorphisms and since $U_1 \cap U_2 = \mathbf{C}\backslash\{0\}$ and $\varphi_2\varphi_1^{-1}$ is a homeomorphism between $\mathbf{C}\backslash\{0\} \to \mathbf{C}\backslash\{0\}$, then $\mathbf{C}_\infty$ is a 2-dimensional topological manifold.

2.3 Regular Points

Let f be a smooth map between manifolds of the same dimension. A point $p \in M$ is regular if the differential T_pf is surjective. As we are only interested in smooth map between Euclidean space of the same dimension, therefore it's desirable to define regular point for M and N to be subsets of $\mathbf{R}^k$ where k is any integer, rather than giving a lengthy discussion over the differential T_pf, it can simply be treated as the derivative df_p of the point p in the Euclidean space. Therefore, a point $x \in \mathbf{R}^k$ is a regular point of f if the derivative df_x is non-singular, while the point $y \in \mathbf{R}^k$ is called a regular value if $f^{-1}(y)$ contains only regular points. In order to study the regular points of the rational functions, we let M = N =

$\mathbf{C}_\infty$ which has dimension 2, therefore, df_p should have rank 2 if p is a regular point. the condition for a complex number to be a regular point can be derived as follows:

Let $z = x + iy$ be a regular point where x and y are real and $f(z) = u(z) + iv(z)$ where u and v are real-valued function, then we have $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ evaluate at p does not vanish and since

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

therefore $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \neq 0$ when evaluated at p which means that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ when evaluated at p do not vanish. As $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ therefore $f'(p) \neq 0$.

On the other hand, those points which are not regular are called *critical points*, therefore, for M to be $\mathbf{S}^2$ or $\mathbf{C}_\infty$, a critical point with finite modulus, x, of f is the point whose derivative vanish, i.e. $f'(x) = 0$ and its image $f(x)$ is called the critical value, if f has a critical point at ∞, then by the discussion in 1.3, we consider $f \circ J$ and evaluate the derivative at the origin which is also vanished but the the critical value is ∞ then we consider $\frac{1}{f(x)}$ which has zero modulus. Let R be a rational function, then R' is also a rational function, and there are finite number of solutions for $R' = 0$ which means that there are finitely many critical points for R. The number of critical points directly affect the nature of the Fatou set.

2.4 Degree of Maps

Now we are going to to define the degree of a map topologically. Let M and N be oriented n-dimensional manifolds and f : M $\rightarrow$ N be a smooth map. Let $x \in$ M be a regular point of f and define the sign of df_x to be +1 and -1 according as df_x preserves or reverses the orientation. For any regular value of $y \in N$, define

$$deg(f\,;y) = \sum \text{sgn } df_x$$

It is known that for any connected space, N, $deg(f; y) = deg(f; y_1)$, where y and y_1 are any points in N. Let us consider the set of all regular values of a rational function $R : \mathbf{C}_\infty \rightarrow \mathbf{C}_\infty$, which is a subset of $\mathbf{C}_\infty$, as we know that there are only a finite number of critical values, therefore it is connected, thus it consist of one component only. Also, it is known that $\mathbf{C}_\infty$ is orientable with $|R'| > 0$ for all regular values, therefore sign $df_x = +1$. As

$$deg(R\,;y) = \sum_{y=R(x)} \text{sgn } dR_x = \# R^{-1}(y),$$

that is the number of preimage of the regular value of y. Since $\mathbf{C}\backslash$\{all critical values\} is connected, therefore,

$$deg(R\,;y) = deg(R\,;y_1) = \#R^{-1}(y),$$

where y_1 is any regular value of R. This agrees with our previous definition of the degree of a rational function $R = \dfrac{P}{Q}$, where P and Q are polynomials with $\#R^{-1}(y)$ is the number of solutions of $P - yQ = 0$ which is the maximum of the $deg(P)$ and $deg(Q)$.

2.5 Euler Characteristics

We devote this section to discuss the Euler Characteristics of S which is a compact subdomain on the complex sphere. The Euler Characteristics is important due to its topological invariant property and it is also an useful tool for classification of topological spaces. Moreover, in the theory of iteration, the importance of Euler Characteristics of S lies on the derivation of Riemann-Hurwitz equation which is crucial in the studying of Fatou set. To define the Euler Characteristics of S, we first partitioned S into a finite number of mutually disjoint subsets called *vertices*, *edges* and *faces* which is called a *triangulation* of S and is denoted by T, with which the edges are homeomorphic to a closed interval $[a, b]$ in R with end points a and b maps to the vertices of T, while the faces are homeomorphic to a closed triangle, F, in C with edges and vertices F map to edges and vertices of a face in T and the interior of F maps to that face. It should be careful that the above definition for faces is valid in a small neighbourhood, for an arbitrary set of S which is compact, we can find a finite number of neighbourhood which covers the given set, thus, faces can be defined for those neighbourhoods and their intersections. We called a vertex, an edge and a face respectively, the *simplex* of *dimension* 0, 1 and 2 of T. The *Euler Characteristics* of a simplex S of dimension m is defined as $(-1)^m$ and is denoted by $\chi(S)$. For any subset S_0 of S which comprise a disjoint union of simplices, S_j where $j = 1, \ldots, n$ with dimension m_j, we define

$$\chi(S_0) = \sum_j \chi(S_j) = \sum_j (-1)^{m_j}$$

Therefore, for surface S, which contains F faces, E edges and V vertices

$$\chi(S) = \sum_j (-1)^{m_j} = F(-1)^2 + E(-1)^1 + V(-1)^0 = F - E + V$$

It's important to note that $\chi(S)$ for any surface is independent of the particular triangulation used, therefore, we can compute $\chi(S)$ by using any convenient triangulation. It is easy to see that if S_1 and S_2 are subsets of S, then

$$\chi(S_1 \cup S_2) = \chi(S_1) + \chi(S_2) - \chi(S_1 \cap S_2)$$

Using the preceeding formula, the calculation of χ can be simplified. For example, by constructing suitable triangulation, it is not hard to see that $\chi(\mathbf{C}_\infty) = 2$, and that for an closed disc D, $\chi(D) = 1$.

Example 2.5.1: The Euler Characteristics can be easily be calculated by introducing a convenient triangulation on the given surface. Let us consider the sphere, $\mathbf{S}^2$ and the disc, $\mathbf{D}$, triangulations of $\mathbf{S}^2$ and the disc are shown in figure 2.1 and 2.2 respectively. The vertices are represented by the dots while the edges can be represented by the arc joining the vertices. The Euler Characteristics so found will be useful in our discussion later.

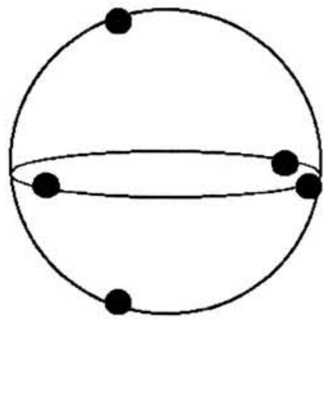


Figure 2.1

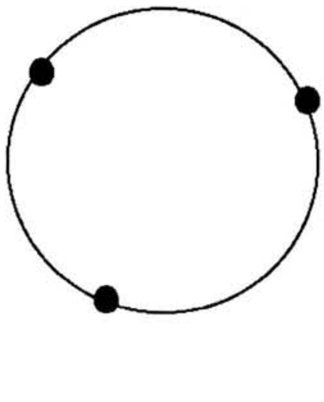


Figure 2.2

The Euler Characteristics of the sphere and the disc are calculated as follows:

$$\chi(\mathbf{S}^2) = 4(-1)^2 + 6(-1)^1 + 4(-1)^0 = 4 - 6 + 4 = 2 \quad \text{and}$$

$$\chi(\mathbf{D}) = 1(-1)^2 + 3(-1)^1 + 3(-1)^0 = 1$$

Now suppose that D is the complex sphere with a circular hole Q on it, then $D \cup Q = \mathbf{C}_\infty$ and $D \cap Q = \varnothing$.

$$\therefore \quad \chi(\mathbf{C}_\infty) = \chi(D) + \chi(Q) - \chi(D \cap Q)$$

$$\chi(D) = 2 - 1 = 1$$

Remark: We will need $\chi(S)$ for non compact subdomain S of $\mathbf{C}_\infty$. However, those S must satisfy the following limiting condition and $\chi(S)$ is still well-defned:

For all sequence of sompact subsets K_i's, $K_1 \subset K_2 \subset \ldots K_n \subset \ldots \subset S$ with $\cup K_j = S$ and $\lim\limits_{j \to \infty} \chi(K_j)$ exists.

2.6 Covering Space

As we may know later that the number of critical points and the number of Fatou components play an important role in the theory of complex dynamics. To count those numbers, we usually use Riemann Hurwritz Formula and we begin by discussing the covering space.

Definition 2.6.1: A map $f : M \to N$ is a *covering* if for all $\omega \in N$, there is an open neighbourhood W of ω and a family of open sets $\{U_i\}_{i \in I}$ such that we have

(1) $U_i \in M$ for all $i \in I$ and $U_i \cap U_j = \varnothing$ for $i \neq j$.

(2) $f|_{Ui} : U_i \to V$ is a homeomorphisms

(3) $f^{-1}(V) = \cup_i U_i$

and M is a *covering space* of N. If, in addition, M is also simply connected, then M is the *universal covering space* of N.

If f is a rational function of degree d and let V be the set of all critical values, and V' be the set of all inverse images of the critical values. Consider $f : \mathbf{C}_\infty \setminus V' \to \mathbf{C}_\infty \setminus V$, it is obvious that f is a covering and we called f, a *branched covering*. Let $M \subseteq \mathbf{C}_\infty$ and $N \subseteq \mathbf{C}_\infty$, then for all $\omega \in N$ and a neighbourhood W of ω we have $U_1, U_2, \ldots, U_d$, a countable number of subsets of M such that (1), (2), (3) of definition 2.6.1 being satisfied and we called f is a *d-fold covering*.

2.7 Riemann Hurwritz Formula

Let $f : M \to N$ be a d-fold covering, where M and N are subsets of $\mathbf{C}_\infty$, we can take an arbitrary triangulation, T on N, then $\chi(N) = V - E + F$ where V, E and F are respectively the vertices, edges and faces of the triangulation, T. From definition 2.6.1, f^{-1} defines a triangulation, T' on M. Clearly, the vertices of T' are inverse images of the vertices in T, and therefore T' has exactly dV vertices. Also each edge and each face of T lifts under each branch of f^{-1} to an edge and a face of T' respectively, and T' has precisely dE adges and dF faces. Now we can deduce that

$$\chi(M) = dV - dE + dF$$

$$= d(V - E + F) = d\chi(N) \qquad \text{equation 2.7.1}$$

From example 2.5.1, we knew that $\chi(\mathbf{S}^2) = 2$ and $\chi(M) = 2 - 1 = 1$ where $M = \mathbf{S}^2 \backslash A$, and A is a circular disc. Now we can construct a sequence of circular discs

$$\{A_n : n = 1, 2, \ldots\} \text{ with } A \supset A_1 \supset A_2 \supset \ldots \supset A_i \supset \ldots$$

where A_i is a circular disc with radius of length half that of A_{i-1}. From the Theorem of the Nested Set, we knew that the sequence will have a limit, ζ. Let $M_i = \mathbf{S}^2 \backslash A_i$, then

$$M \subset M_1 \subset M_2 \subset M_3 \ldots$$

Also $\chi(M) = \chi(M_1) = \chi(M_2) = \chi(M_3) = \ldots = 2 - 1 = 1$

which is a constant with value, 1 and therefore for the limiting case,we have

$$\chi(\mathbf{S}^2 \backslash \{\zeta\}) = 2 - 1 = 1$$

Inductively, we have a relation

$$\chi(\mathbf{S}^2 \backslash \{\zeta_1, \zeta_2, \ldots, \zeta_n\}) = 2 - n, \text{ where } \zeta_i\text{'s are points on } \mathbf{S}^2$$

Given $R : U \to V$ is a rational map of degree d, let $\omega_1, \omega_2, \ldots, \omega_q$ be critical values and let the preimages of ω_i be $R^{-1}(\omega_i) = \{z_{i,1}, z_{i,2}, \ldots, z_{i,p_i}\}$. As we know from any standard text of complex analysis that

$$R(z) = \omega_i + (z - z_{i,1})^{\upsilon_{i,1}} \cdots (z - z_{i,p_i})^{\upsilon_{i,p_i}} h_i(z),$$

where $h_i(z) \neq 0$, $\upsilon_{i,j}$ are the valencies of R at $z_{i,j}$ also $\sum_{j=1}^{p_i} \upsilon_{i,j} = d$. In other words, we have q critical values and their pre-images consists of $\sum_{i=1}^{q} p_i$ points. Of course, for each critical value ω_i , at least one of the pre-image $z_{i,j}$ is critical, that is $R'(z_{i,j}) = 0$ and we have $\upsilon_{i,j} > 1$.

Let $U' = U \setminus \{z_{i,j} : j = 1,\dots, p_i;\ i = 1,\dots, q\}$ and $V' = V \setminus \{\omega_i : i = 1,\dots, q\}$. That is, we remove everything related to a critical value. Then $R : U' \to V'$ is a d-fold covering.

Thus by $$\chi(U') = \chi(U) - \sum_{i=1}^{q} p_i \quad \text{and} \quad \chi(V') = \chi(V) - q.$$

Also by equation 2.7.1, $$\chi(U') = d\chi(V')$$

that is
$$\chi(U) - \sum_{i=1}^{q} p_i = d(\chi(V) - q)$$

$$\chi(U) + dq - \sum_{i=1}^{q} p_i = d\chi(V)$$

$$\chi(U) + \sum_{i=1}^{q} (d - p_i) = d\chi(V) \sum_{i=1}^{q} \sum_{j=1}^{p_i} (\upsilon_{i,j} - 1)$$

$$\chi(U) + \sum_{i=1}^{q} \sum_{j=1}^{p_i} (\upsilon_{i,j} - 1) = d\chi(V)$$

It should be noted that if $z_{i,j}$ is not a critical point, then $\upsilon_{i,j} - 1 = 0$ otherwise $\upsilon_{i,j} - 1 \geq 1$, also the term $\sum_{j=1}^{p_i} (\upsilon_{i,j} - 1)$ is defined as the *deficiency* of R at the critical value ω_i and is denoted by $\delta_R(\omega_i)$ and that $\sum_{i=1}^{q} \sum_{j=1}^{p_i} (\upsilon_{i,j} - 1)$ is the total deficiency of R over V, which is denoted by $\delta_R(V) = \sum \delta_R(\omega)$ for all $\omega \in V$. Therefore we get Riemann-Hurwritz Formula:

$$d\chi(V) = \chi(U) + \sum_{\omega \in V} \delta_R(\omega)$$

Chapter 3 The Montel Theorem

3.1 Introduction

As we know from the development of theory of iteration, the study of complex dynamics began in the late nineteenth century which originally relied principally on results developed internally, such as fixed point theorems and theorems pertaining to the solutions of various functional equation and with little relation with complex function theory. Until the second decade of the twentieth century. The French Mathematicians, Fatou and Julia, study the topic by utilizing important theorems from the theory of complex functions. In their studies, they both divided $\mathbf{C}_\infty$ into regions of normality and non-normality, which are often called, respectively the Fatou sets ad the Julia sets (Details will be discussed in later chapters). By the application of Montel's theory of normal families, a global approach of describing iteration of arbitrary complex functions beyond the neighbourhood of a fixed point can be done. This approach, at that time, is a fresh and innovative one in the study of iteration and is a stepping stone in the modern study of complex dynamics. In this chapter, we are going to discuss the Montel's theorem of normal families of functions.

3.2 Normality and Equicontinuous

Definition 3.2.1: A family $\aleph$ of maps on a domain Ω is equicontinuous at $x_0 \in \Omega$, if and only if for every positive ε there exists a positive δ such that for all $x \in \Omega$ and for all $f \in \aleph$

$$|x_0 - x| < \delta \text{ implies } |f(x_0) - f(x)| < \varepsilon$$

The family $\mathcal{N}$ is *equicontinuous* on a subset X_0 of Ω if it is equicontinuous at each point x_0 of X_0.

As seen from definition 3.2.1, every function in $\mathcal{N}$ maps the open ball with centre x_0 and radius δ into a ball of radius at most ε, that means, if $\mathcal{N}$ is equicontinuous on X, the action of any element of $\mathcal{N}$ on two points in X which are closed together will have some control on the distance between the two images, i.e. the images will be not be too far apart.

The concept of equicontiuity is closely related to normal families of analytic function, with the results of normal family of analytic functions, we can derive further results in the theory of iteration.

Definition 3.2.2: A family $\mathcal{N}$ of analytic functions on a domain $\Omega \subseteq \mathbf{C}$ is *normal* in Ω. If every sequence of functions $\{f_n\} \subseteq \mathcal{N}$ contains a subsequence which converges to a limit function f uniformly on each compact subset of Ω.

It's known from Weierstrauss Theorem that the limit function is an analytic function. the family $\mathcal{N}$ is said to be normal at a point $z_0 \in \Omega$ if it is normal in some neighbourhood of z_0. It should be noted that the limit function need not lie in $\mathcal{N}$. As we mentioned before: equicontinuity and normality are closely related and their relation can be expressed by Arzela -Ascoli Theorem, this is proved in many texts(see [1]) and we omit the proof.

THEOREM 3.2.3: Let D be a subdomain of the complex sphere and $\aleph$ be a family of continuous maps of D into the complex sphere. Then $\aleph$ is equicontinuous in D if and only if it is a normal family in D.

From the above theorem, we noted that equicontinuity and normality are two equivalent concepts in the complex sphere.

3.3 Local Boundedness

The concept of local boundedness of families of functions which can be seen as a generalization of boundedness of a function plays an important role in our discussion of normality.

Definition 3.3.1: A family of function $\aleph$ is *locally bounded* on a domain Ω if, for each $z_0 \in \Omega$, there is a positive number M and a neighbourhood $D(z_0; r) \subseteq \Omega$ such that $|f(z)| \leq M$ for all $z \in D(z_0; r)$ and all $f \in \aleph$.

To illustrate the concept of local boundedness, we consider the following examples:

Example 3.3.2: Any family $\aleph$ of functions into the unit disc

$$\mathbf{D} = \{\omega \in \mathbf{C} : |\omega| < 1\}$$

is locally bounded by taking any compact neighbourhood K, for any point z_0 and $M = 1$. the family, in fact, is globally bounded.

Example 3.3.3: Let's consider the family $\aleph$ which is defined as

$$\{f : f(z) = z^r \text{ and } 0 < r \in \mathbf{Q}\}.$$

It's not locally bounded if the domain of the function is **C**. Since by taking $z_0 = 1$, a compact neighbourhood K always contain a point $1 + \varepsilon$ with a small $\varepsilon > 0$ and $\sup_K |z^r|$ is not bounded for $0 < r \in \mathbf{Q}$. While it is locally bounded if the domain of the function is **D**. In this case we have all the functions map into **D** and we take $M = 1$.

It is important to note that the condition of local boundedness depends not only on the family of functions but also on the domain in which they were defined.

THEOREM 3.3.4: If $\aleph$ is a family of locally bounded analytic function on a domain Ω, then the family of derivatives $\aleph' = \{f' : f \in \aleph\}$ form a locally bounded family in Ω.

Proof : For any $z_0 \in \Omega$, there is a closed neighbourhood $K(z_0; r) \subseteq \Omega$ and a constant M such that $|f(z)| \le M$, $z \in K(z_0; r)$. Then for $z \in D(z_0; r/2)$ and $\zeta \in C_r = \{z : |z - z_0| = r\}$, the Cauchy Formula gives

$$|f'(z)| \le \frac{1}{2\pi} \int_{C_r} \frac{|f(\zeta)||d\zeta|}{|\zeta - z|^2} < \frac{4M}{r}$$

for all $f' \in \aleph'$, so that $\aleph'$ is locally bounded.

There is a relationship between local boundedness and equicontinuity which is stated as the following theorem:

THEOREM 3.3.5: A locally bounded family $\aleph$ of analytic functions on a domain Ω is equicontinuous on compact subset of Ω.

Proof: By theorem 3.3.4, $\aleph$ is locally bounded implies $\aleph' = \{f' : f \in \aleph\}$ is uniformly bounded on compact subsets of Ω. For a closed disk $K \subseteq \Omega$, we have $|f'(z)| \leq M$ for all $z \in K, f' \in \aleph'$, and some constant M. Then for any two points $z, z' \in K$, integrating over a straight line path from z to z' gives

$$|f(z) - f(z')| \leq \int_{z'}^{z} |f'(\zeta)||d\zeta| \leq M|z - z'|$$

Hence, given $\varepsilon > 0$ and choosing $0 < \delta < \varepsilon / M$,

$$|f(z) - f(z')| < \varepsilon$$

whenever $z, z' \in K$, $|z - z'| < \delta$. therefore $\aleph$ is equicontinuous on K.

Let E be a compact set in Ω, each point of E is the centre of a closed disk with same radius r which form a covering of E, then we can select a finite subcover $S_1, S_2, \ldots, S_n$, also, given any ε, there is a $\delta' = \min(\delta, r)$ such that for any two points z and z' which satisfies the relation

$$|z - z'| < \delta'$$

then they must lie in the same disk S_r or they must lie on a larger disk with radius $2r$, since the family of closed disk with centre at each point of E and radius $2r$ form a covering of E, by compactness, there is a finite subcover $C_1, \ldots, C_m$ with centre at $z_1, \ldots, z_m$ respectively, if $|z - z'| < \delta$, then z lies in C_i and

$$|z' - z_1| < |z' - z| + |z - z_1| < r + r = 2r$$

which means that z' also lies in C_i, by the result of the above, they are equicontinuous. therefore, $\mathcal{N}$ is equicontinuous on a compact subset of Ω.

By Ascoli and Arzela Theorem, we can restate Theorem 3.3.5 as:

THEOREM 3.3.6: If a family $\mathcal{N}$ of analytic functions in **C** is locally bounded then it is normal.

3.4 Covering and Uniformization

There are several very important techniques using covering spaces which had been defined in section 2.6. In this section, we are going to bind together all the concepts of topological spaces, their covering spaces ad manifolds, we will only discuss the concepts and the theorems are stated with the proofs omitted.

Recall from the discussion in section 2.6, if $M \to N$ is a *covering space*, they are locally the same but many neighbourhoods in M map to one in N. Thus, a local Euclidean representation for one is also valid for another, that is, M and N are simultaneously manifolds of the same kind. Moreover, it should be noted that, the covering map is an analytic map between the two manifolds.

Let $M \to N$ be a covering space of N, if M is simply connected then it is called a *universal covering space* of N. Universal cover always exists and is unique in the class of manifolds.

THEOREM 3.4.1 (Uniformization Theorem): Let N be an open subset of $\mathbf{C}_\infty$, the universal cover is

- $\mathbf{C}_\infty = \mathbf{S}^2$ if $N = \mathbf{C}_\infty = \mathbf{S}^2$
- $\mathbf{C}$ if $N = \mathbf{C}$ or $\mathbf{C}\backslash\{\text{one point}\}$
- D(the unit disc) for all other $N \subset \mathbf{C}_\infty$

The first two cases of the theorem may be called the Riemann Mapping Theorem. In our study, we will mostly use the third case and we will discuss it in more details. Before we carry on, we state two more theorems which are useful in our discussion.

THEOREM 3.4.2: Let $N \subset \mathbf{C}_\infty$ and $\wp: M \to N$ be its universal cover (so they may only be $\mathbf{C}$ or D). If ω_n is a sequence in N which converges to $\omega_\infty \notin N$, then for any sequence of its pre-image in the universal cover, i.e., $z_n \in M$ with $\wp(z_n) = \omega_n$, we always have z_n tends to the boundary of M(i.e., if $M = \mathbf{C}$, it is the north; if M = D, it's the unit circle).

Suppose we have a map $f: S \to N$ from a manifold S. Fix ζ_0, i.e. $\omega_0 = f(\zeta_0) \in N$, there are many pre-images of ω_0 in $\wp^{-1}\omega_0$. Fix one of these pre-images and call it u_0.

THEOREM 3.4.3: If S is simply connected, then there is a map $\widetilde{f}: S \to M$ such that $\widetilde{f}(\zeta_0) = u_0$ and $\wp \circ \widetilde{f} = f$. In addition, the map $\widetilde{f}$ preserves the good properties of f (continuous, smooth, analytic, ... etc.).

We will only use the case with S, M, N are domains in $\mathbf{C}_\infty$ and f, $\widetilde{f}$ are analytic. In some book, the above theorem is referred to as analytic continuation or Monodromy Theorem.

3.5 Montel's Theorem

There are several versions of the theorem and we may need the following version. Let $\mathcal{N}$ be a family of analytic functions from a domain $D \subset \mathbf{C}_\infty$ to $\mathbf{C}_\infty$ (some authors may call it meromorphic when the target includes ∞).

THEOREM 3.5.1: If all functions in $\mathcal{N}$ omit three distinct points in $\mathbf{C}_\infty$ then $\mathcal{N}$ is normal.

The three points can be assumed to be 0, 1, ∞. Otherwise, if the points are a, b and c, we may consider the following change for every function f (and this does not alter the normality). Define

$$g(z) = \frac{c-a}{c-b} \cdot \frac{f(z)-b}{f(z)-a}.$$

Since when all functions omit ∞, they are actually analytic functions in $\mathbf{C}$. Therefore, we may restate theorem 3.5.1 as:

THEOREM 3.5.2: A family $\mathcal{N}$ of analytic functions into $\mathbf{C}\backslash\{0, 1\}$ is normal.

Proof: Let $\mathcal{N}$ be a family of analytic functions from D into $\mathbf{C}\backslash\{0,1\}$. Firstly, since if a family is normal on every open disk in D, then it is normal in D, without loss of generality, we may assume that D is a disk, $\mathbf{D}$. Secondly, we construct a new family, by noting that the universal cover of $\mathbf{C}\backslash\{0,1\}$ is $\mathbf{D}$ and for $f \in \mathcal{N}$, $f : \mathbf{D} \to \mathbf{C}\backslash\{0, 1\}$. Take a fixed point ζ_0 in $\mathbf{D}$. For each $f \in \mathcal{N}$, choose a point in $\wp^{-1}(f(\zeta_0))$ and obtain an analytic continuation $\tilde{f} : \mathbf{D} \to \mathbf{D}$. Therefore, we have a new family of functions $\mathfrak{F}$ formed by these analytic functions $\tilde{f}$. From the result of example 3.3.2. $\mathfrak{F}$ is locally bounded, therefore $\mathfrak{F}$ is normal.

Take an arbitrary sequence f_n in $\mathcal{N}$, since $\mathfrak{F}$ is normal, we consider the corresponding sequence $\tilde{f}_n$ which has a subsequence converges uniformly on compact subsets of $\mathbf{D}$. Let the convergent subsequence be $\tilde{f}_{n_k}$ and now we will consider the corresponding f_{n_k}. It is obvious that $f_{n_k}(\zeta_0)$ is a convergent subsequence of $f_n(\zeta_0)$ and let the limit of the subsequence be $\alpha \in \mathbf{C}_\infty$. To accomplish the proof, we have to consider 4 cases:

(a) $\alpha \in \mathbf{C}\backslash\{0, 1)$

(b) $\alpha = 0$

(c) $\alpha = 1$

(d) $\alpha = \infty$

This together with the fact that $\tilde{f}_{n_k}$ has a limit, F, we can show that $\mathcal{N}$ is normal.

(a) For $\alpha \neq 0, 1, \infty$. Take a compact subset $\widetilde{K}$ of **D** such that $K = \wp(\widetilde{K})$ contains α. Note that K is compact subset of **C**\{0, 1}, since

$$\left\| f_{n_k} - \wp \circ F \right\| = \left\| \wp \circ \widetilde{f}_{n_k} - \wp \circ F \right\| \leq M \left\| \widetilde{f}_{n_k} - F \right\|$$

due to Mean Value Theorem and $\wp'$ is bounded on compact set $\widetilde{K}$. As $\widetilde{f}_{n_k}$ converges to F, therefore f_{n_k} converges to $\wp \circ F$.

(b) If f_n is a sequence of function from **D** to **C**\{0, 1} that $f_{n_k}(\zeta_0)$ converges to 1. Then we consider the 2-fold cover $\omega \to \omega^2$: **C** → **C**. This 2-fold cover sends **C**\{0,1, -1} to **C**\{0, 1}. And each function f_n has 2 'lifts' determined by

$$g_n(\zeta_0) = -\sqrt{f_n(\zeta_0)} \quad \text{or} \quad h_n(\zeta_0) = \sqrt{f_n(\zeta_0)}$$

We consider g_n, which satisfies $g_{n_k}(\zeta_0)$ converges to $-1 \neq 0, 1, \infty$. Apply this g_{n_k} in the previous proof and we know that g_{n_k} converges uniformly on compact subsets of **D**. Clearly, $f_{n_k} = {g_{n_k}}^2$ also converges uniformly.

(c) If $f_{n_k}(\zeta_0)$ converges to 0, we may take $g_n = 1 - f_n$ which is still a sequence of functions from **D** to **C**\{0, 1} and $g_{n_k}(\zeta_0)$ converges to 1. This becomes the preceding case and the uniform convergence of g_{n_k} and f_{n_k} on the compact subsets are clearly equivalent.

(d) If $f_{n_k}(\zeta_0)$ converges to ∞, we take $g_n = 1/f_n$ which is still a sequence of functions from **D** to **C**\{0, 1} and $g_{n_k}(\zeta_0)$ converges to 0, and by a similar argument as (c), f_{n_k} converges uniformly on compact subsets of **D**, and this completes the proof.

Chapter 4 Fatou Set and Julia Set

4.1 Iterations of functions

To study complex dynamics, we applied repeatedly a function on a complex variable which we called *iteration* of functions. For a rational function, R, to be iterated for n times, we denote it by $R \circ R \circ \cdots \circ R(z)$ and we simplify the notation by writing it as $R^n(z)$. It is natural to ask if $z_n = R^n(z_0)$, does $\{z_n\}$ tends to a limit point when n tends to infinity? Which initial points upon iteration converge and which do not? For those points which converge upon iterations, we may want to know: for points which were closed to them, do they also converge? Before answering these questions, we define the *orbit* of a point z which is the set of forward iterates $\{z, R(z), R^2(z), \ldots\}$. The answers of the above questions may be obtained by considering the orbit of the points. In the following examples, we shall study the orbits of a particular point for a given rational function.

EXAMPLE 4.1.1: Let $R(z) = z^2$ and the initial point $z_1 = 0.995 + 0.01i$, $z_2 = 1 + 0.01i$ and $z_3 = \cos 29° + i\sin 29°$, we plot the orbits of the three points by iterating each point for a number of times and the orbits are shown through a gradual change of both colour and size, i.e., when n is increasing, the size of the corresponding point plotted become smaller and its colour become darkened, it was noted from Figure 4.1 and 4.2 that $R^n(z_1)$ and $R^n(z_2)$ converges to the origin and ∞ respectively while from Figure 4.3, the orbit of z_3 shows chaotic behaviour, it wandered along the unit circle and we cannot predict its behaviour when n is large.

Figure 4.1 The orbit of $0.995 + 0.01i$ by iterating it with $f(x) = x^2$ for 10 times

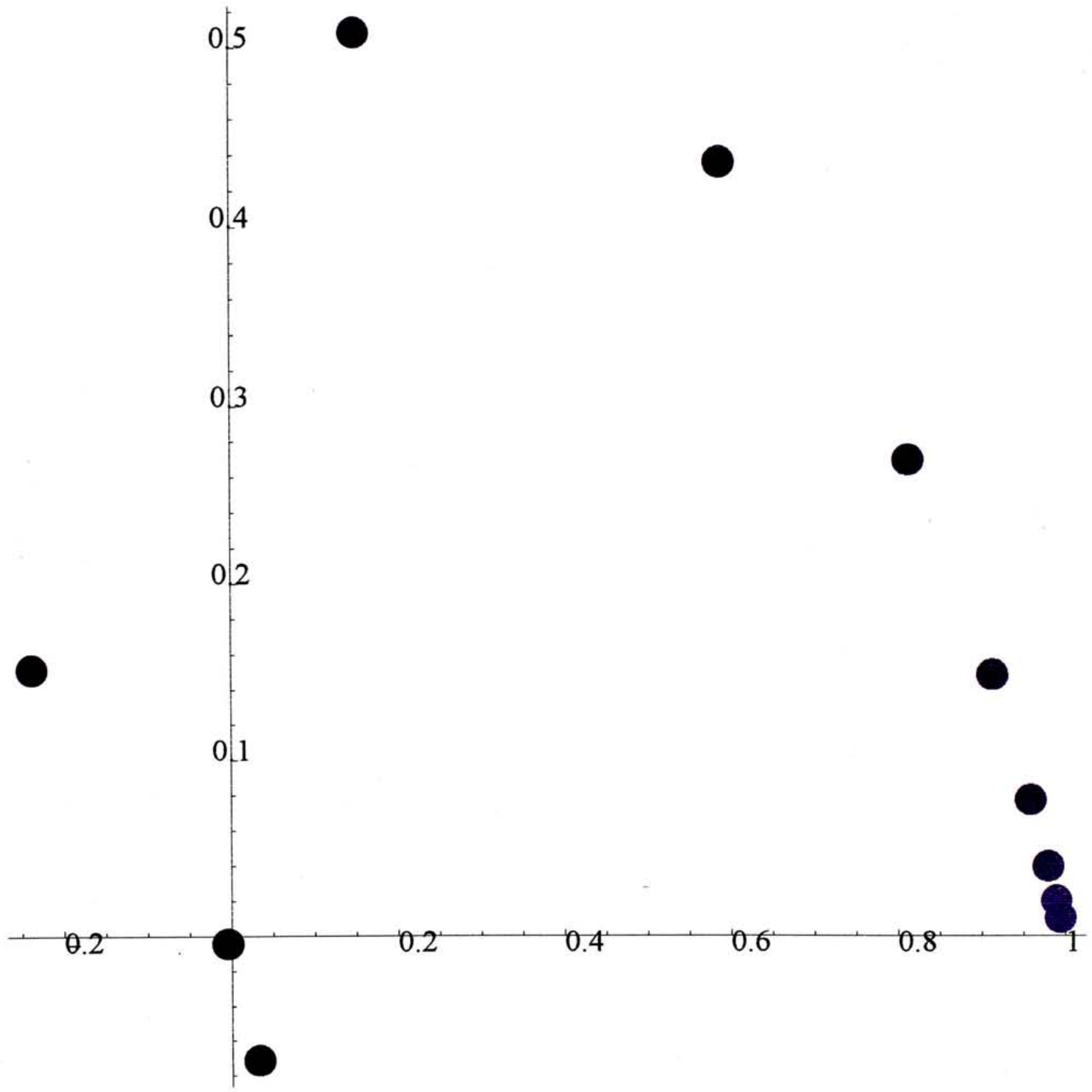


Figure 4.2 The orbit of $1 + 0.01i$ by iterating it with $f(x) = x^2$ for 15 times

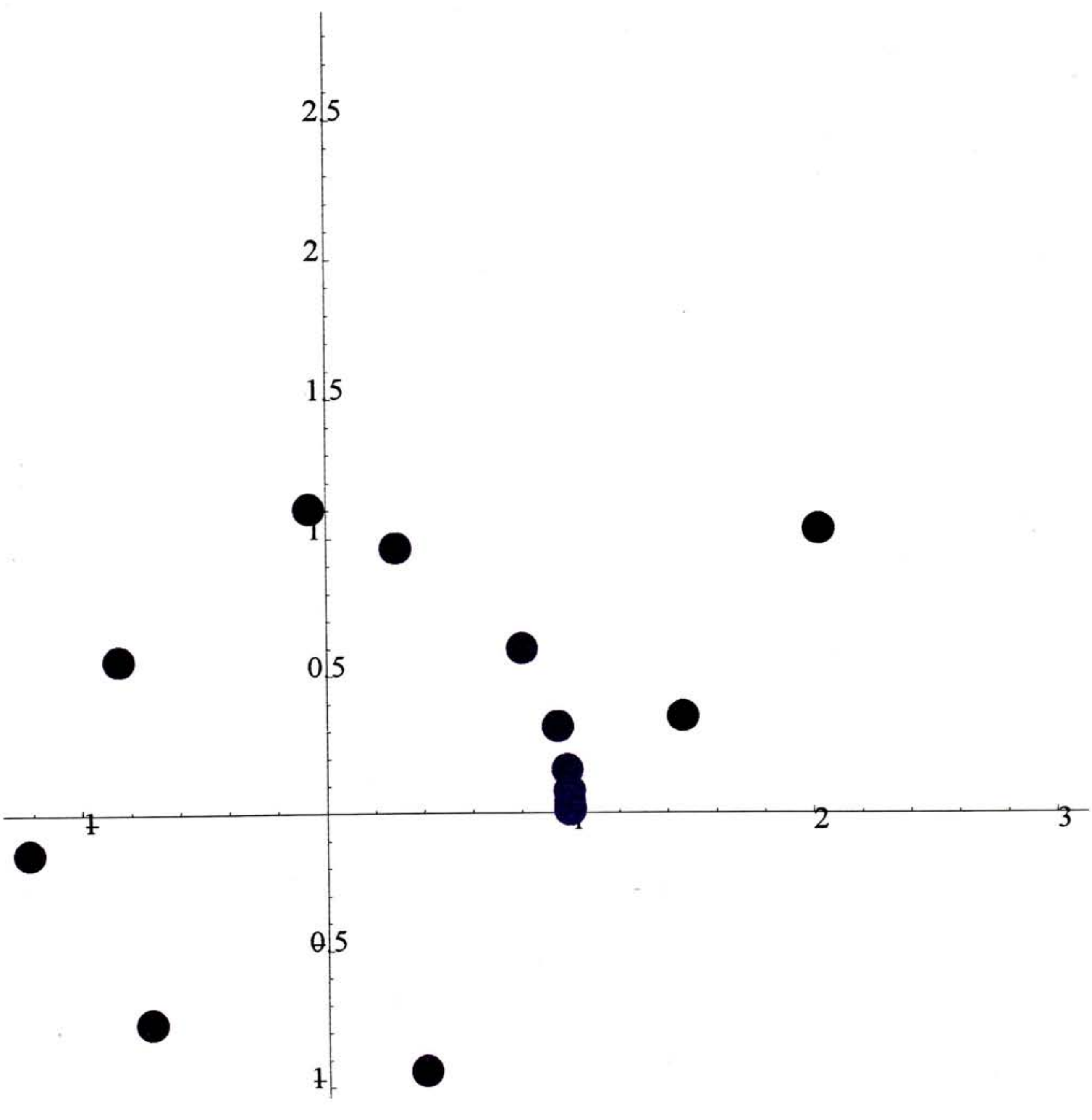
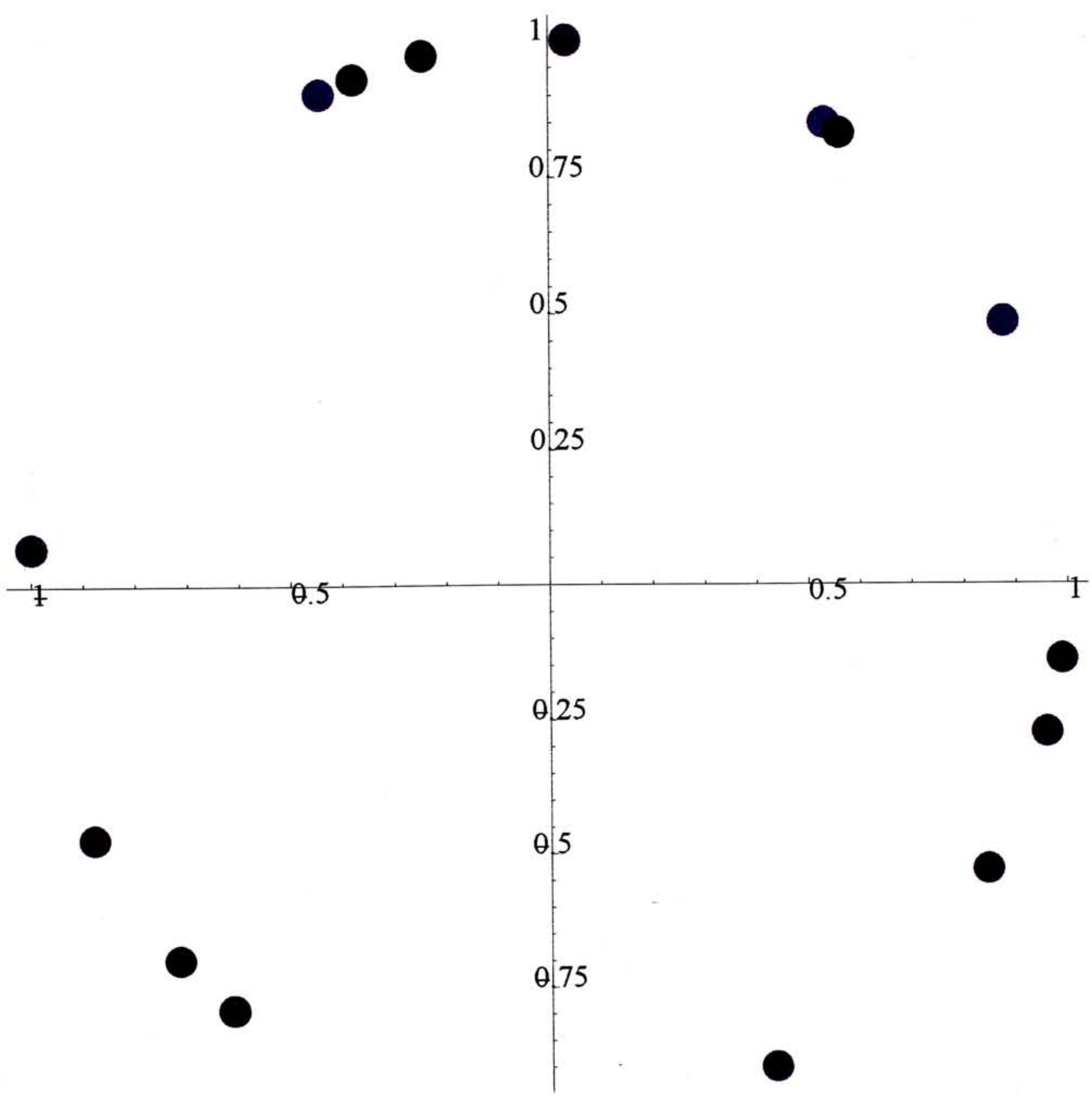


Figure 4.3 The orbit of $\cos 29° + i \sin 29°$ by iterating it with $f(x) = x^2$ for 20 times



From example 4.1, it is natural to think that points on the complex plane may be divided into two categories: the set containing those points which converge upon iteraton and those which do not.

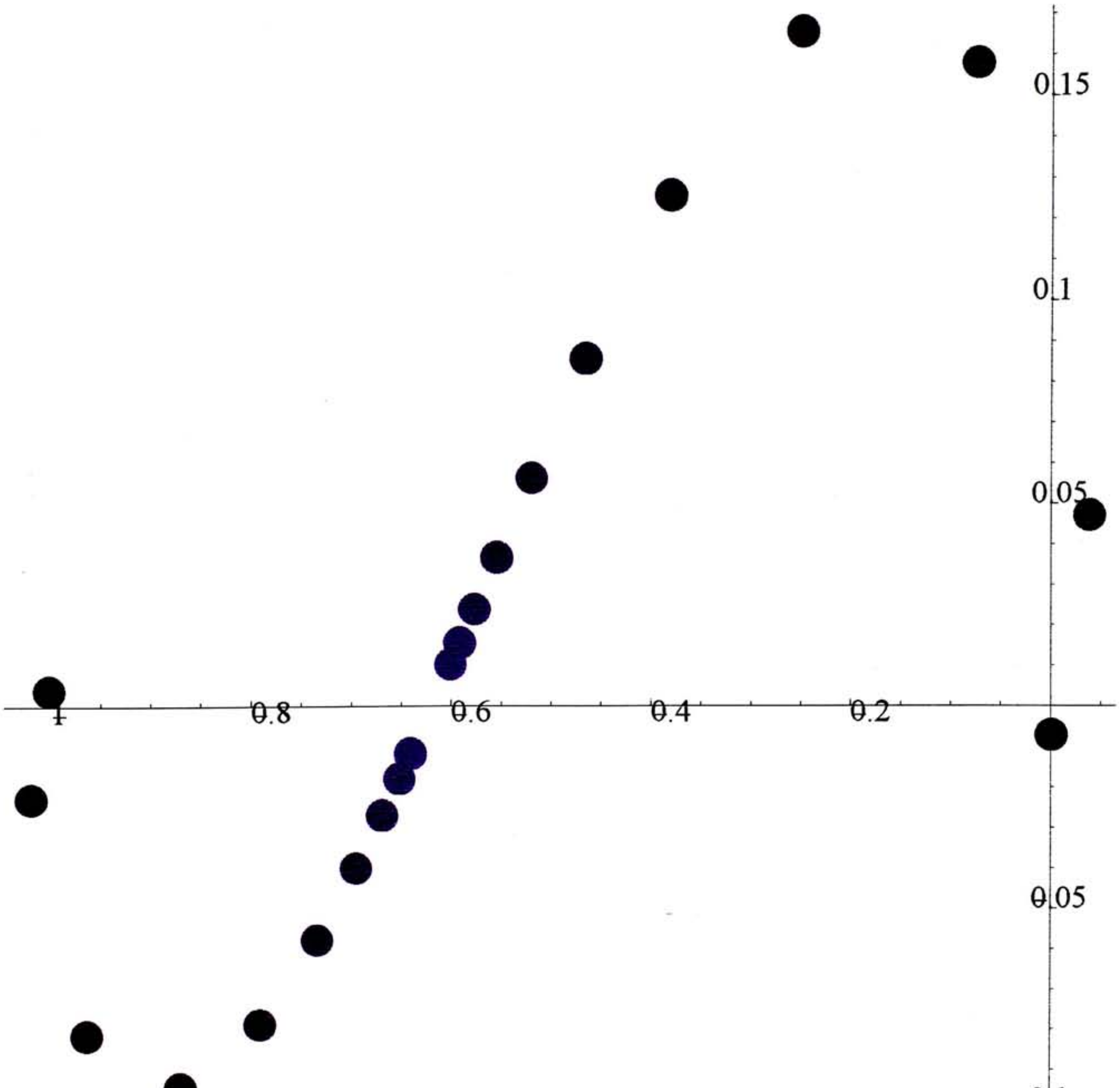
EXAMPLE 4.1.2: Let $R(z) = z^2 - 1$ and take $z_0 = -0.6 + 0.01i$, we plot the orbit of z_0 in Figure 4.4 and found from the figure that $R^n(z_0)$ does not converge but it can be observed that the orbit can be divided into two groups with one of which tending to 0 and the other tending to -1, i.e. a subsequence of it does converge.

Therefore, given a rational function R, the extended complex plane can be partitioned into two sets denoted by F and J which are called Fatou set and Julia set respectively. The iterates preserve proximity on F (and usually converge) whereas this action on J is chaotic.

4.2 Fatou Set and Julia Set

DEFINITION 4.2.1: The *Fatou set* F of a rational function R, which sometimes avoid confusion may denoted by $F(R)$, is defined to be the set of points $z_0 \in \mathbf{C}_\infty$ such that $\{R^n\}$ is a normal family in some neighbourhood of z_0.

Figure 4.4 The orbit of -0.6+0.01i by iterating it with $f(x) = x^2 - 1$ for 20 times



As we know from the result of Arzela-Ascoli that a family $\mathcal{F}$ is normal on some neighbourhood D if and only if it is equicontinuous there. Therefore we have another definition for Fatou set which is equivalent to the above definition.

DEFINITION 4.2.2: The Fatou set F of R is the maximal open subset of $\mathbf{C}_\infty$ on which $\{R^n\}$is equicontinuous.

While the *Julia set*, J of R is the complement of the Fatou set in $\mathbf{C}_\infty$. It is easy to see from the definition that F is open and J is compact. From previous dicussion, we know that conjugation can sometimes simplified the process of iteration and it should not be surprised to note that the Fatou set of a rational function and its conjugate are related.

THEOREM 4.2.3: Let R be a non-constant rational map and g be a Mobius map where $S = gRg^{-1}$, then $F(S) = g(F(R))$ and $J(S) = g(J(R))$.

PROOF: Take any element $x_0 \in F(S)$ and consider the family $\{S^n\}$, since $x_0 \in F(S)$, therefore for any $\varepsilon > 0$, there is a $\delta \in \mathbf{R}$ such that $\forall\, x \in F(S)$, if $|x - x_0| < \delta$, then

$$|S^m(x) - S^m(x_0)| < \varepsilon$$

As $S^m = gR^m g^{-1}$, therefore $|gR^m g^{-1}(x) - gR^m g^{-1}(x_0)| < \varepsilon$

Let $g(y) = x$ and $g(y_0) = x_0$, then,

when $|y - y_0| < \delta$, $|gR^m(y) - gR^m(y_0)| < \varepsilon$, since g is a Mobius Transformation.

As g^{-1} is also a Mobius map, then there is a ε_1, such that

$|gR^m(y) - gR^m(y_0)| < \varepsilon$, then $|g^{-1}(gR^m(y)) - g^{-1}(gR^m(y_0))| < \varepsilon_1$, that is

$$|R^m(y) - R^m(y_0)| < \varepsilon_1,$$

Therefore for $y_0 \in F(R)$ then $g^{-1}(x_0) \in F(R)$ that is $x_0 \in g(F(R))$.

Take $x_0 \in g(F(R))$, then $g^{-1}(x) \in F(R)$, $\therefore$ for any ε,

there is a δ such that $\forall\ g^{-1}(x) \in F(R)$

$$|g^{-1}(x) - g^{-1}(x_0)| < \delta \Rightarrow |R^m g^{-1}(x) - R^m g^{-1}(x_0)| < \varepsilon$$

since g is a mobius map, therefore, there is a $\varepsilon_1 > 0$ such that

$$|gR^m g^{-1}(x) - gR^m g^{-1}(x_0)| < \varepsilon_1$$

that is $\quad |S^m(x) - S^m(x_0)| < \varepsilon_1$

i.e. $x_0 \in F(S)$

Take $\quad x \in J(S)$

$\Leftrightarrow x \in \mathbf{C} \setminus J(R)$

$\Leftrightarrow x \in \mathbf{C} \setminus g(F(R))$

$\Leftrightarrow x \in \mathbf{C}$ and $x \notin g(F(R))$

$\Leftrightarrow x \in \mathbf{C}$ and $g^{-1}(x) \in J(R)$

$\Leftrightarrow x \in \mathbf{C}$ and $x \in g(J(R))$

$\Leftrightarrow x \in g(J(R))$

THEOREM 4.2.4: For any non-constant rational map R, and any positive integer p, $F(R^P) = F(R)$ and $J(R^p)=J(R)$

PROOF: Let $S = R^p$. First, as $\{S^n : n \geq 1\}$, it is equicontinuous wherever $\{R^n : n \geq 1\}$ is, therefore $F(R) \subseteq F(S)$. Next, consider the family

$$\mathcal{F}_k = \{R^k S^n : n \geq 0\} \quad \text{where } k \in \mathbf{N}$$

It is equicontinuous on the Fatou Set $F(S)$ of S, as each R^k is uniform continuous,

$\therefore\ \forall\ \varepsilon \in \mathbf{R}$, there is $\delta_1 \in \mathbf{R}$ such that $|y - y_0| < \delta_1 \Rightarrow |R^k(y) - R^k(y_0)| < \varepsilon$

since for any $x_0 \in F(S)$ and for this ε_1, there is a $\delta \in \mathbf{R}$ such that

$$\forall\ x \in \mathrm{F}(S),\ |x - x_0| < \delta \Rightarrow |S^n(x) - S^n(x_0)| < \delta_1$$
$$\Rightarrow |R^k S^n(x) - R^k S^n(x_0)| < \varepsilon$$

It is also easy to see that the finite union $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{p\text{-}1}$ is equicontinous on $F(S)$

As this union is $\{R^n : n \geq 0\}$, the family $\{R^n : n \geq 1\}$ is equicontinuous on $F(S)$ and so $F(S) = F(R)$. i.e. $F(R^p) = F(R)$

4.3 Iteration of Mobius Transformations

Before the discussion of the Fatou set of rational function, in general, it is good for us to study some easy cases. The iteration of Mobius Transformation which is of degree one should be the simplest. We begin by considering two examples:

EXAMPLE 4.3.1: Let $R(z) = \dfrac{4z-3}{3z-2}$. It is easy to verify that $R(1) = 1$, i.e. 1 is the fixed point of R, we now choose $0.9 + 0.001i$ and 0.95 which is closed to 1 and study the orbits (as shown in Figure 4.5), we find that the iterates z_n start to move away from the point 1 but eventually return to a neighbourhood of 1, actually, it converges to 1.

EXAMPLE 4.3.2: Let $R(z) = e^{i\frac{\pi}{6}}\left(\dfrac{10z-1}{z-10}\right)$. Figure 4.6 shows 3 orbits by iterating R on 3 different initial points, $0.5 + 0.5i$, 1 and $1.5 - 0.2i$ for 20 times, we can see that $R^n(z)$ lie on a circle. It was also noted that 1 and ∞ are fixed points for R and no convergence was observed.

Figure 4.5 The orbits of 0.9 +0.001i and 0.95 by iterating them with

$$R(z)=\frac{4z-3}{3z-2} \text{ for 20 times}$$

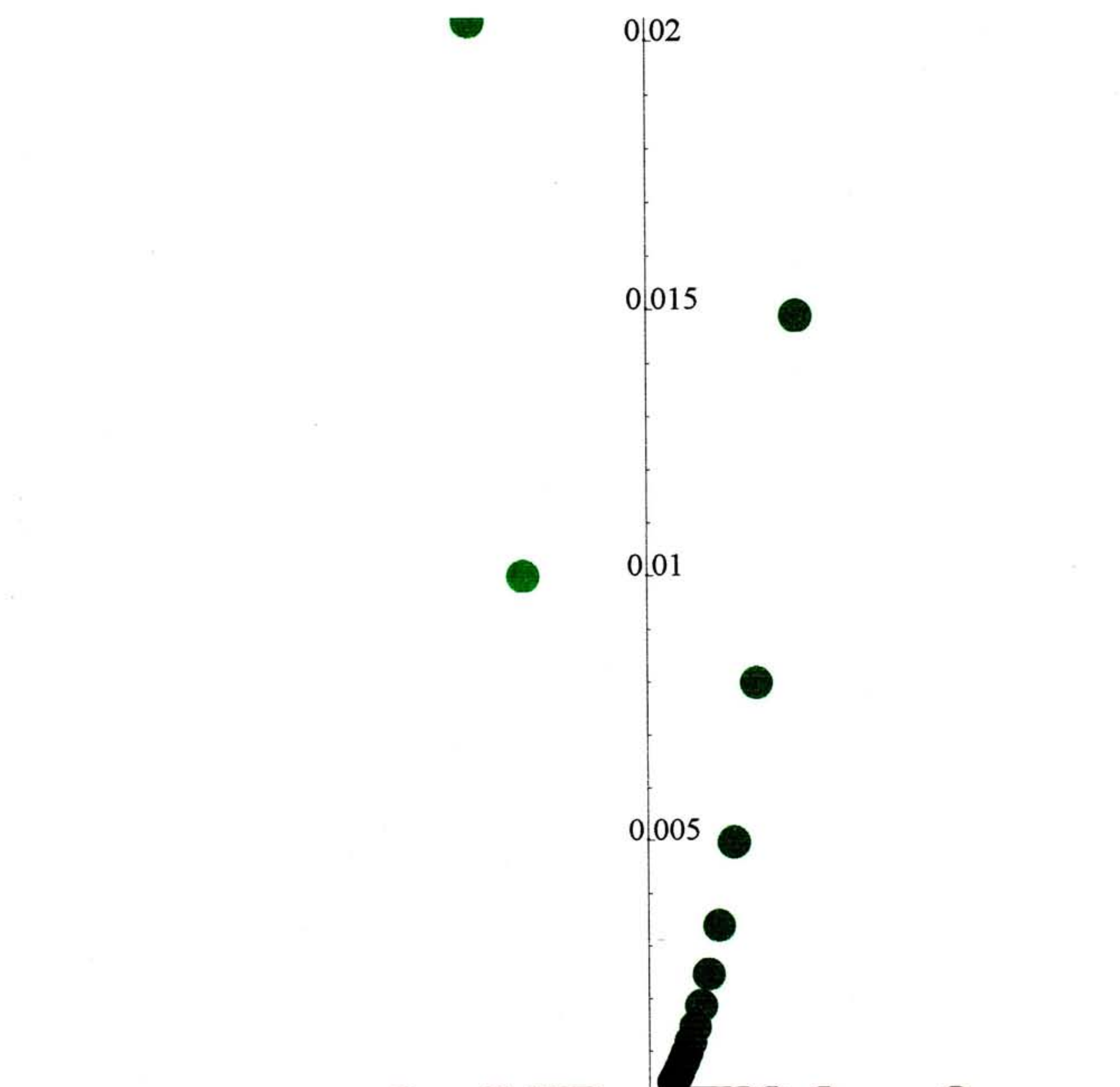
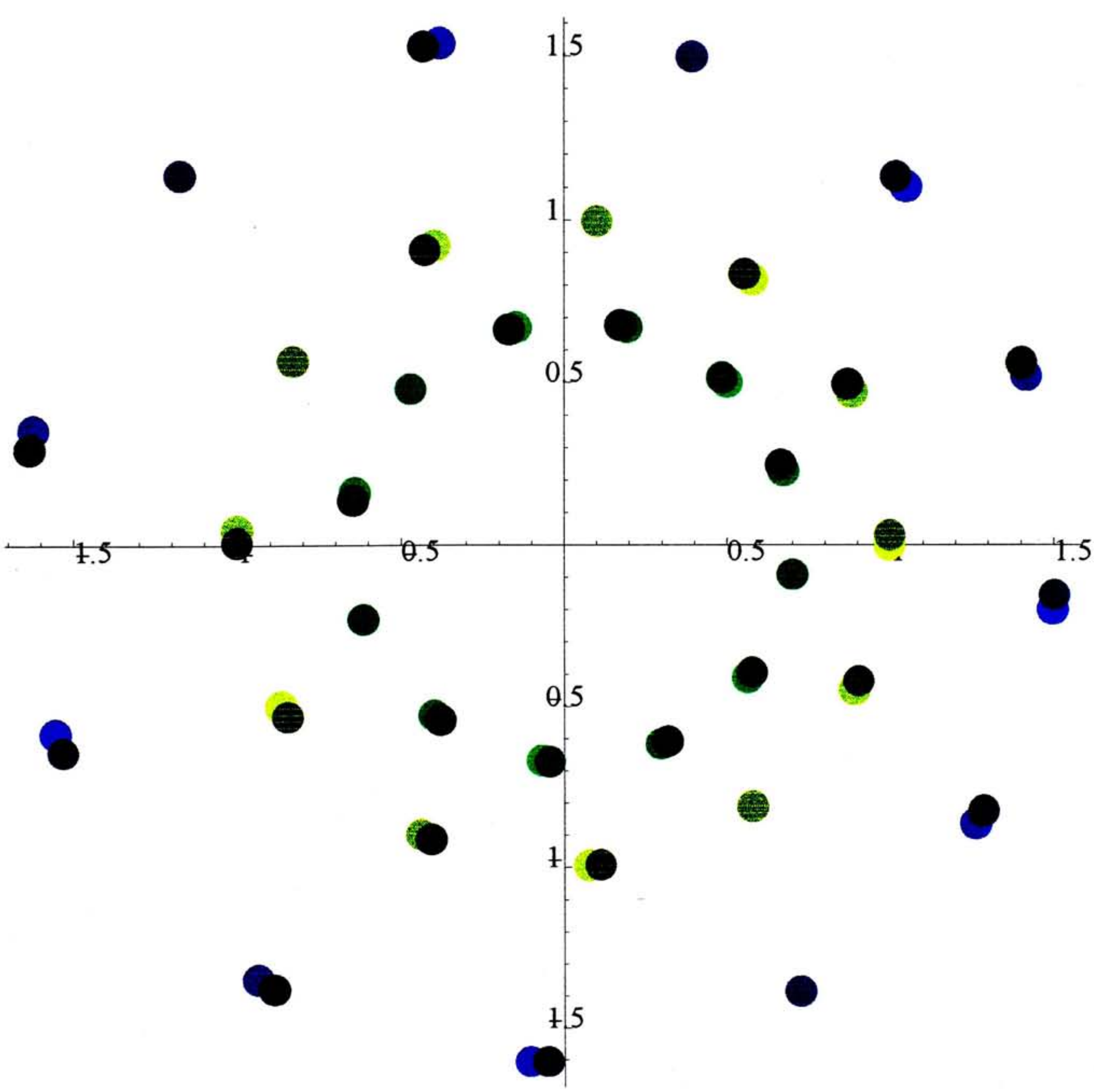


Figure 4.6 The orbits of $0.5+0.5i$, 1 and $1.5-0.2i$ by iterating them with

$$R(z) = e^{i\frac{\pi}{6}}\left(\frac{10z-1}{z-10}\right) \text{ for 20 times}$$



In the above discussions, we should note that any Mobius Transformations have only one or two fixed ponts and have no other periodic points, therefore we may expect some underlying simple results in the theory of the iteration of rational function of degree one. Let's begin by studying their fixed point:

For Mobius Transformation, f, with one fixed point, ζ, we find another Mobius Transformation, μ such that

$$\mu(z) = \frac{1}{z-\zeta}.$$

Consider the conjugate of f, say g, which can be expressed as $\mu^{-1}f\mu$, note that g fixes ∞ and be expressed explicitly as $g(z) = z + a$ where $a \neq 0$, so $g^n(z) = z + na$ and for every z, $g^n(z) \to \infty$ as $n \to \infty$, since $f^n = \mu g^n \mu^{-1}$, therefore $f^n(z) \to \mu(\infty) = \zeta$, it is interesting to note that ∞ is a neutral fixed point for g since $|g'| = 1$ and ζ is also a neutral fixed point for f. As seen from example 4.3.1, we shall expect that some points may have an orbit which moves towards ζ and some points may have an orbit which start by moving away from ζ and eventually converge to ζ.

For Mobius Tranformations, f, having two distinct fixed points ζ_1 and ζ_2, we take

$$\mu(z) = \frac{z-\zeta_1}{z-\zeta_2}$$

and find the conjugate of f which is expressed as $g = \mu f \mu^{-1}$, it is obvious that g fixes 0 and ∞ and g can be expressed explicitly as $g(z) = kz$, therefore, $g^n(z) = k^n z$. The value of k governs the properties of the iteration, as the number of iterations tend to ∞ and there are four cases:

(1) If $|k| < 1$, then $g^n(z) \to 0$, that is $f^n(z) \to g^{-1}(0) = \zeta_1$.

(2) If $|k| > 1$, then $g^n(z) \to \infty$, that is $f^n(z) \to g^{-1}(\infty) = \zeta_2$

(3) If $|k| = 1$ and k is an n^{th} root of unity, then the orbit of any point is a finite set of points on some circle.

(4) If $|k| = 1$ and k is not a root of unity, then points in the orbit of any point form a dense subset of some circle.

We have now shown that in the case of Mobius transformation, it gives a rather simple theory which covers all circumstances under the iteration of rational function of degree one, and from now on, we shall only be concerned with theory of iteration of rational functions of degree at least two.

4.4 Fixed Points and their Classification

A fixed point z of a rational function R is the point which satisfies the equation $R(z) = z$. To study the complex dynamical systems, it is desirable to begin with the description of local behaviour near the fixed points. Let z_0 be a fixed point of R, suppose z is closed to z_0, then, approximately

$$|R(z) - z_0| = |R(z) - R(z_0)| = |R'(z_0)|\,|z - z_0| \qquad \text{(EQUATION 4.4.1)}$$

From equation 4.4.1, we notice that the value of $|R'(z_0)|$ which we called the *multiplier* of R at z_0, is important in determining the properties of the orbits of the points which are closed to z_0 upon applying R. Suppose $|R'(z_0)| < 1$, then points closed to z_0 move even closer to it when applying R, therefore we can classify the fixed point according to the value of $|R'(z_0)|$.

DEFINITION 4.4.1: Suppose that $z_0 \in \mathbf{C}$ is a fixed point of a rational function R. Then z_0 is

(1) a *superattracting* fixed point if $R'(z_0) = 0$

(2) an *attracting* fixed point if $|R'(z_0)| < 1$

(3) a *repelling* fixed point if $|R'(z_0)| > 1$

(4) a *rationally neutral* fixed point if $|R'(z_0)| = 1$ and $R'(z_0)$ is a root of unity

(5) an *irrationally neutral* fixed point if $|R'(z_0)| = 1$ but $R'(z_0)$ is not a root of unity.

It is easy to see that points which are closed to a repelling fixed point tend to move away from it when applying R. On the other hand, if z lies sufficiently close to an attracting fixed point z_0, then $R^n(z) \to z_0$ as $n \to \infty$. Also it is not easy to predict the behaviour of the points in the neighbouhood of a neutral point when we apply R. From the definition above, it is noted that a super-attracting fixed point is also a critical point of R.

4.5 Periodic Points and Cycles

DEFINITION 4.5.1: A point z_0 is a *periodic point* of a rational function R if it is a fixed point of some iterate R^n, that is $R^n(z_0) = z_0$ where n is a positive integer while

$$\{z_0, R(z_0), \ldots, R^{n-1}(z_0)\}$$

has n distinct elements which is called the *cycle* of z_0 and the integer n is the *period* of z_0. In particular, the fixed points of R are points of period one. Generally speaking, z_0 has period n if and only if it is a fixed point of R^n but not of any lower-order iterate. Let z_0 be a fixed point of R^n and we may assume that the cycle $\{z_0, R(z_0), \ldots, R^{n-1}(z_0)\}$ does not contain ∞ through conjugation. By applying chain rule for n times.

$$\begin{aligned}\left(R^n\right)'\left(z_0\right) &= \prod_{k=0}^{n-1} R'\left(R^k\left(z_0\right)\right)\\ &= R'(z_0)\cdot R'\left(R(z_0)\right)\cdots R'\left(R^{n-1}(z_0)\right)\end{aligned}$$

The above equation shows that the derivative $(R^n)'$ has the same value at each point $R^m(z_0)$ of the cycle, since

$$\begin{aligned}\left(R^n\right)'\left(R^m\left(z_0\right)\right) &= \prod_{k=0}^{n-1} R'\left(R^k\left(R^m\left(z_0\right)\right)\right)\\ &= \prod_{k=0}^{n-1} R'\left(R^{m+k}\left(z_0\right)\right)\\ &= \prod_{k=0}^{n-1} R'\left(R^k\left(z_0\right)\right)\end{aligned} \qquad \text{(Equation 4.5.1)}$$

while the third product in equation 4.5.1 being a re-arrangement of the second. Therefore each point on a cycle can be classified in exactly the same way as any other point in the cycle, consequently, we can define naturally *the multiplier of a cycle* which is $\prod_{k=0}^{n} R'(R^k(z_0))$, where $\{R^k(z_0), k = 0, 1\ldots, n-1\}$ is a cycle and classify the cycle as a super attracting cycle, attracting cycle and so on according to the modulus of the multiplier of the cycle.

Thus for an attractive fixed point $z_0 \in \mathbf{C}_\infty$, we are led to define the set

$$A(z_0) = \{z \in \mathbf{C}_\infty : R^n(z) \to z_0 \text{ as } n \to \infty\},$$

called the *basin(domain) of attraction* of z_0. The *immediate basin of attraction* of z_0, $A^*(z_0)$ is the connected component of $A(z_0)$ containing z_0. For a periodic orbit of period n,

$$\gamma = \{z_0,\ z_1 = R(z_0),\ \ldots,\ z_{n-1} = R^{n-1}(z_0)\},$$

then the immediate basin of attraction of γ is given by

$$A^*(\gamma) = \bigcup_{i=0}^{n-1} A^*(z_i, R^n),$$

where $A^*(z_i, R^n)$ is the immediate attractive set for the attractive fixed point z_i of the mapping R^n. The immediate basin of attraction of a cycle is important since it has a close relation with the Julia set and will be discussed later.

4.6 Critical Points

A point z_0 is a *critical point* of a non constant rational map R if R fails to be injective in any neighbourhood of z_0, these are the points with valencies greater than 1, since the valency of z_0, k, is determined by the condition that the limit

$$\lim_{z \to z_0} \frac{R(z) - R(z_0)}{(z - z_0)^k}$$

exists, where it is also finite and non-zero. For critical points, they have valencies greater than one, therefore $\lim_{z \to z_0} \dfrac{R(z) - R(z_0)}{z - z_0} = 0$, since R is a rational map and it is analytic throughout $\mathbf{C}$ which means that $R'(z_0) = 0$ and this agrees with the previous discussion on the condition which determine the critical point, we shall see that the number of critical points for a rational map play an important role in iteration theory, therefore we are going to discuss the estimation of the number of critical points.

We know from the previous section that if R is an m-fold map of U onto V, then by Riemann-Hurwitz formula $\chi(U) + \delta_{R(U)} = m\chi(V)$, where $\delta_{R(U)}$ is the total deficiency of R over U, $\chi(U)$ and $\chi(V)$ are respectively the Euler Characteristics of U and V. By taking U and V to be $\mathbf{C}_\infty$, we have

$$\chi(U) = \chi(V) = 2 \text{ and } m = deg(R) \text{ then } \delta_{R(\mathbf{C}\infty)} = 2m - 2$$

As $\delta_{R(\mathbf{C}\infty)} = \Sigma[\upsilon_{R(z)} - 1]$ where $\upsilon_{R(z)}$ is the valency of z on R, we know that all points on $\mathbf{C}_\infty$ have valency one except for the critical points, therefore a rational map of positive degree d has at most $2d - 2$ critical points in $\mathbf{C}_\infty$, if multiplicity of the critical point is count, then a rational map of degree d has exactly $2d - 2$ critical points.

4.7 Illustrations of local behaviour of map near periodic points

The local behaviour of the map near the periodic points can be illustrated by considering of the orbits of points in a neighbourhood of the periodic points. Perhaps, the idea may be more concrete by considering a few examples.

EXAMPLE 4.7.1: Consider $R(z) = z^2 - \frac{z^3}{9}$, it is not hard to check that the origin is the super attracting fixed point and 1.145898... is a repelling fixed point. In Figure 4.7, we show the orbits of iterating R on -0.9 + 0.3i, and 0.65 for a number of times and can be obseved that the orbits converge quickly to the origin. When we iterate R on 1.15 + 0.01i and 1.15 − 0.02i for 5 times, the orbits show a completely different picture, they went away from 1.145898..., as shown in Figure 4.8.

EXAMPLE 4.7.2: Let $R(z) = \dfrac{2z}{z+1}$ then the origin is a repelling fixed point and 1 is an attracting fixed point, we plot the orbits of iterating each of -0.1 + 0.1i, -0.1 − 0.1i, 0.1 − 0.1i and 0.1 for 5 times and they were shown in Figure 4.9, a rather similar phenomenon as in the previous example can be observed, the orbits seemed to diverge from the origin which is a repelling fixed point and as we plot the orbits with the same initial points, z_0 but iterating them for 10 times, it can be observed that the orbits converges to 1 which is the attracting fixed point as shown in Figure 4.10. This example match quite well with the discussion in section 4.3.

By now, we can draw some conclusions from the observations of the previous examples: In a small neighbourhood of a fixed point, the direction of motion of the orbits by iterating the points inside the neighbourhood is towards the attracting fixed point while that of a repelling fixed point is away from the repelling fixed point. In the following examples, we consider the cases of a neutral fixed point.

Figure 4.7 The orbits by iterating -0.9 + 0.3i and 0.65 with $R(z) = z^2 - \frac{z^3}{9}$ for 8 times and 5 times respectively.

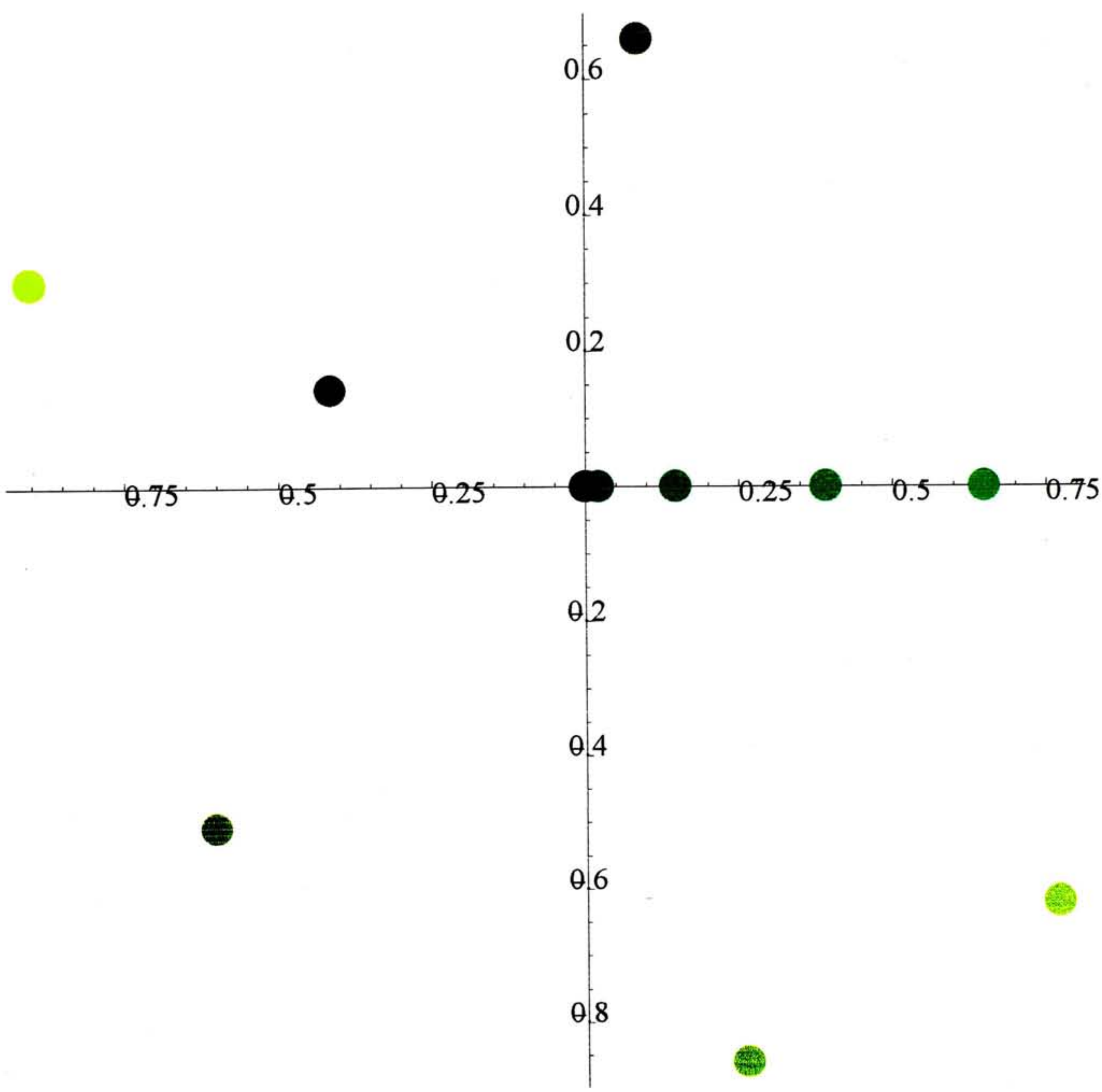


Figure 4.8 The orbits by iterating $1.15 + 0.01i$ and $1.15 - 0.02i$

with $R(z) = z^2 - \frac{z^3}{9}$ for 5 times.

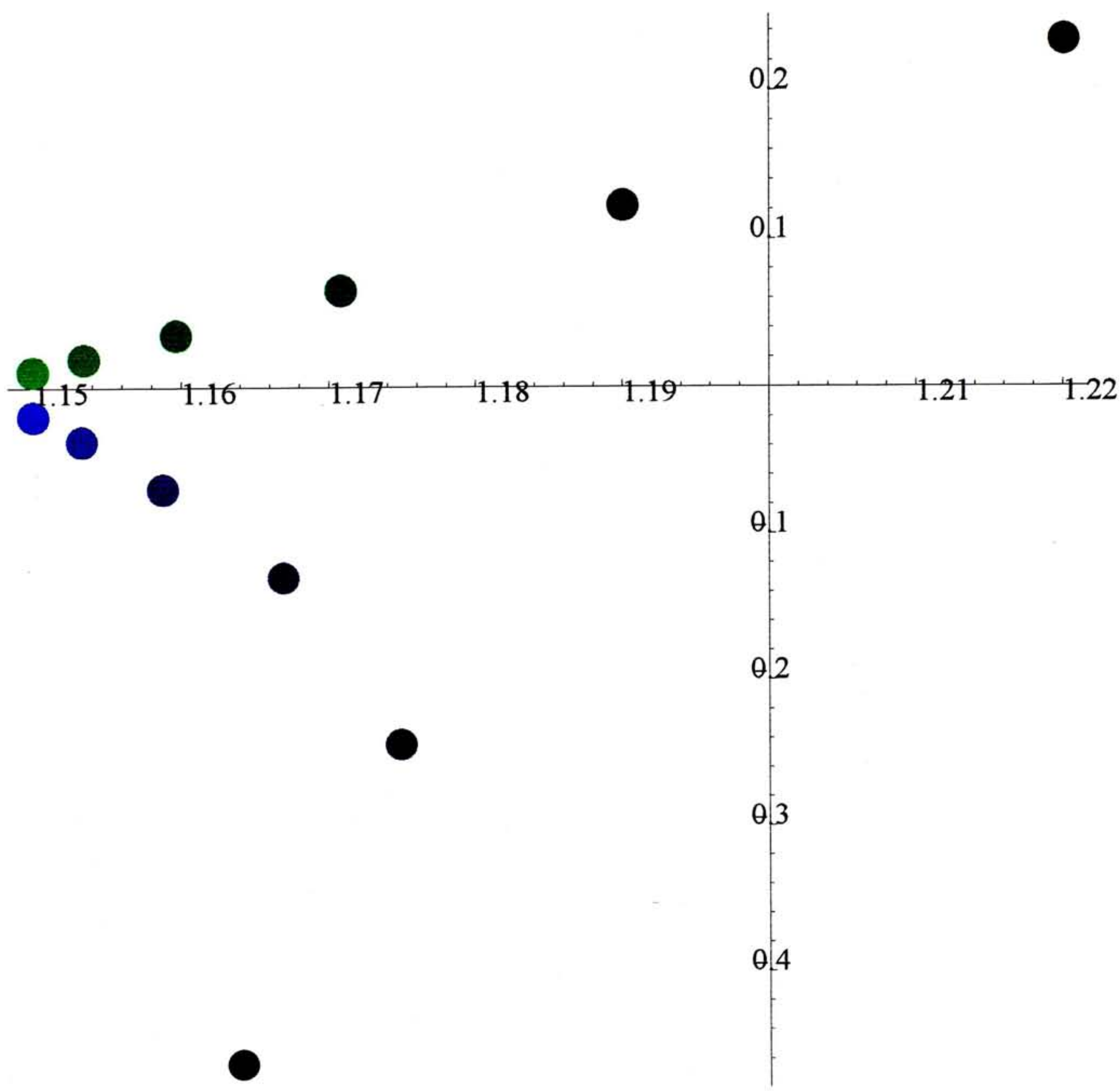


Figure 4.9 The orbits by iterating -0.1 + 0.1i, -0.1 – 0.1i, 0.1 – 0.1i and 0.1 with $R(z) = \dfrac{2z}{z+1}$ for 5 times.

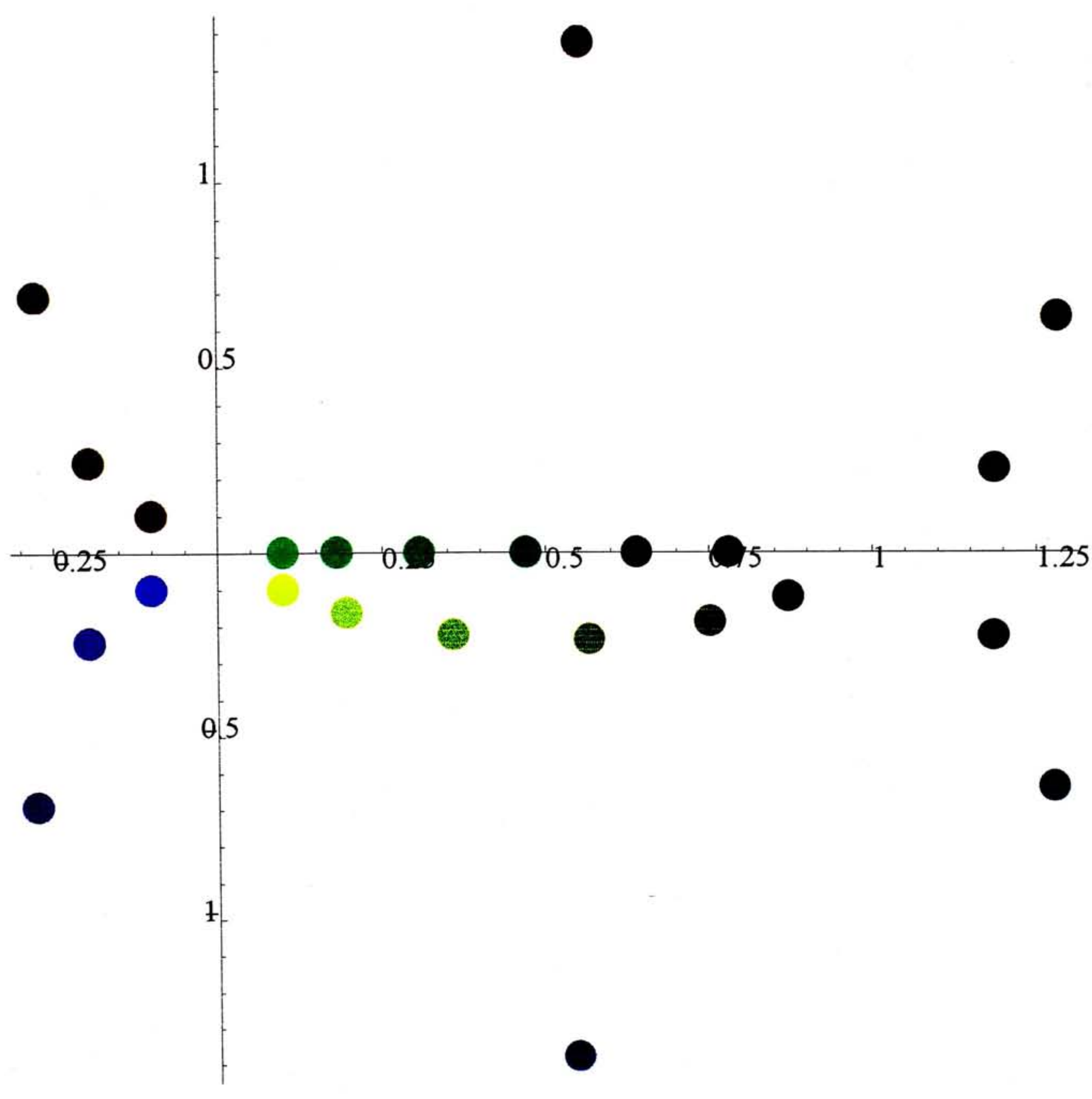
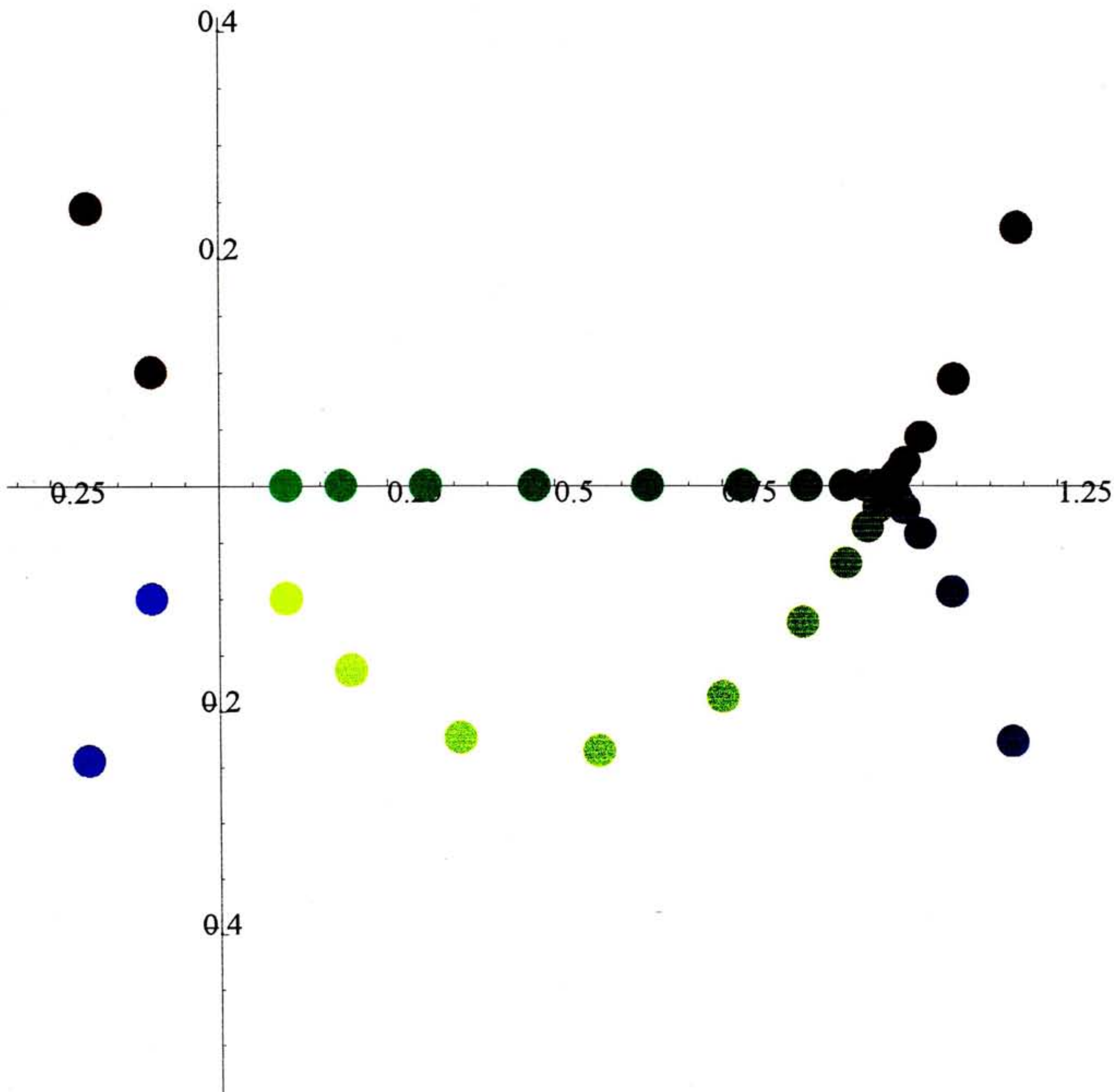


Figure 4.10 The orbits by iterating -0.1 + 0.1i, -0.1 – 0.1i, 0.1 – 0.1i and 0.1

with $R(z) = \dfrac{2z}{z+1}$ for 10 times.



EXAMPLE 4.7.3: It is easy to see that the origin is a neutral fixed point of $z + z^4$, we plot the orbits by iterating each of the points $0.25 - 0.01i$, $-0.5 + 0.01i$, $0.5 + 0.4i$, and $-0.4 - 0.1i$ for 20 times, we observe from Figure 4.11 that the direction of motion of the orbits differed from cases in previous examples, not all orbits move towards or away from the fixed point but some move towards and some away from it.

EXAMPLE 4.7.4: The origin is also the neutral point of $z - 4z^7 + 6z^{10} - 6z^{13} + z^{16}$, when we plot the orbits by iterating each of the points 0.5, -0.5, $0.3i$ and $-0.3i$ for 200 times, we have a similar observation and the orbits are shown in Figure 4.12.

We conclude from the observations that the neighbourhood of a neutral fixed point shows a different behaviour from that of the attracting and repelling fixed point. The direction of motion of the orbits were not either towards or away from the fixed point , but with some of them move towards and some away from the fixed point, and the direction of motion depends on the position of the initial point z_0.

Figure 4.11 The orbits by iterating $0.25 - 0.01i$, $-0.5 + 0.01i$, $0.5 + 0.4i$ and $-0.4 - 0.1i$ with $R(z) = z + z^4$ for 20 times.

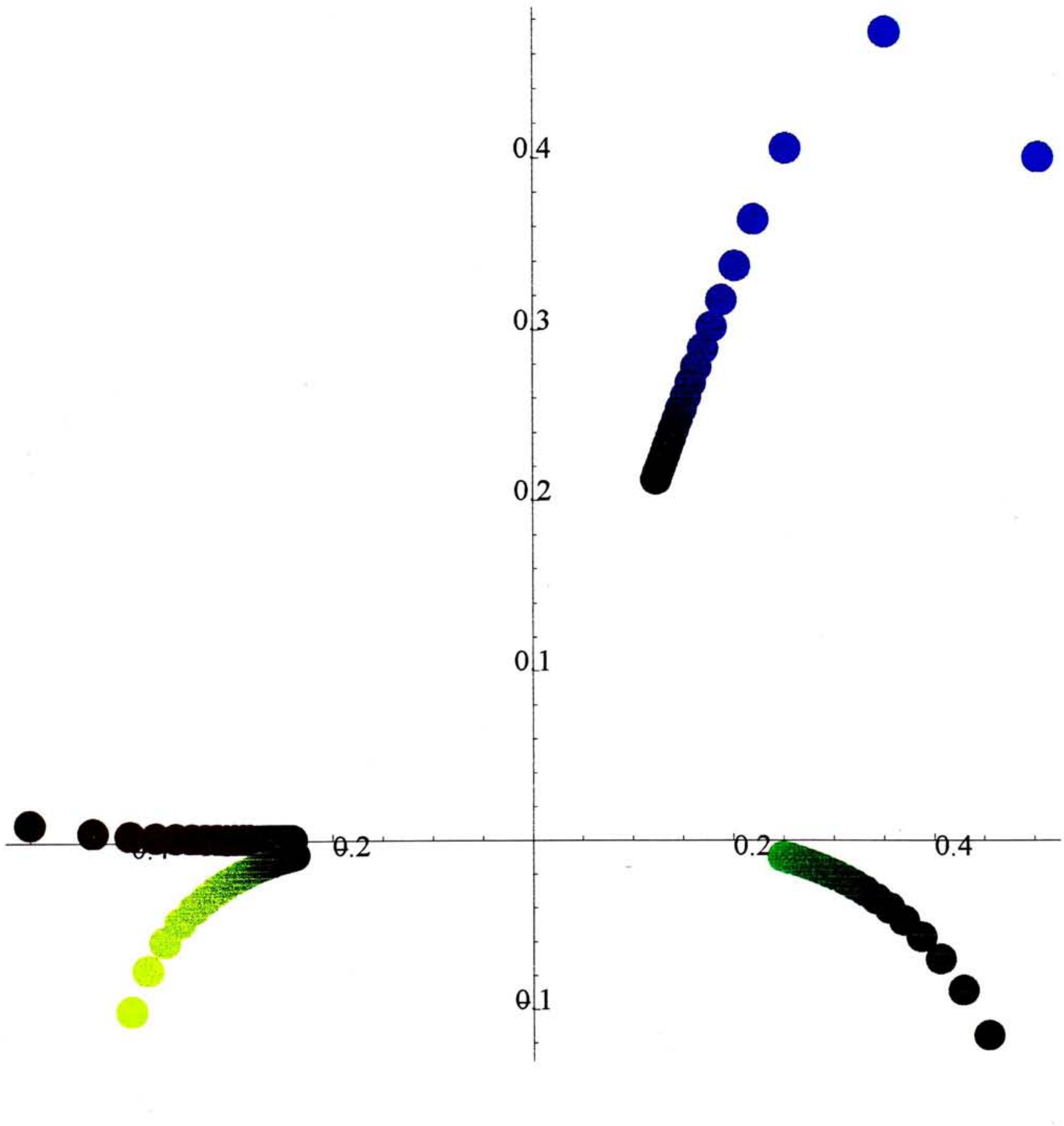
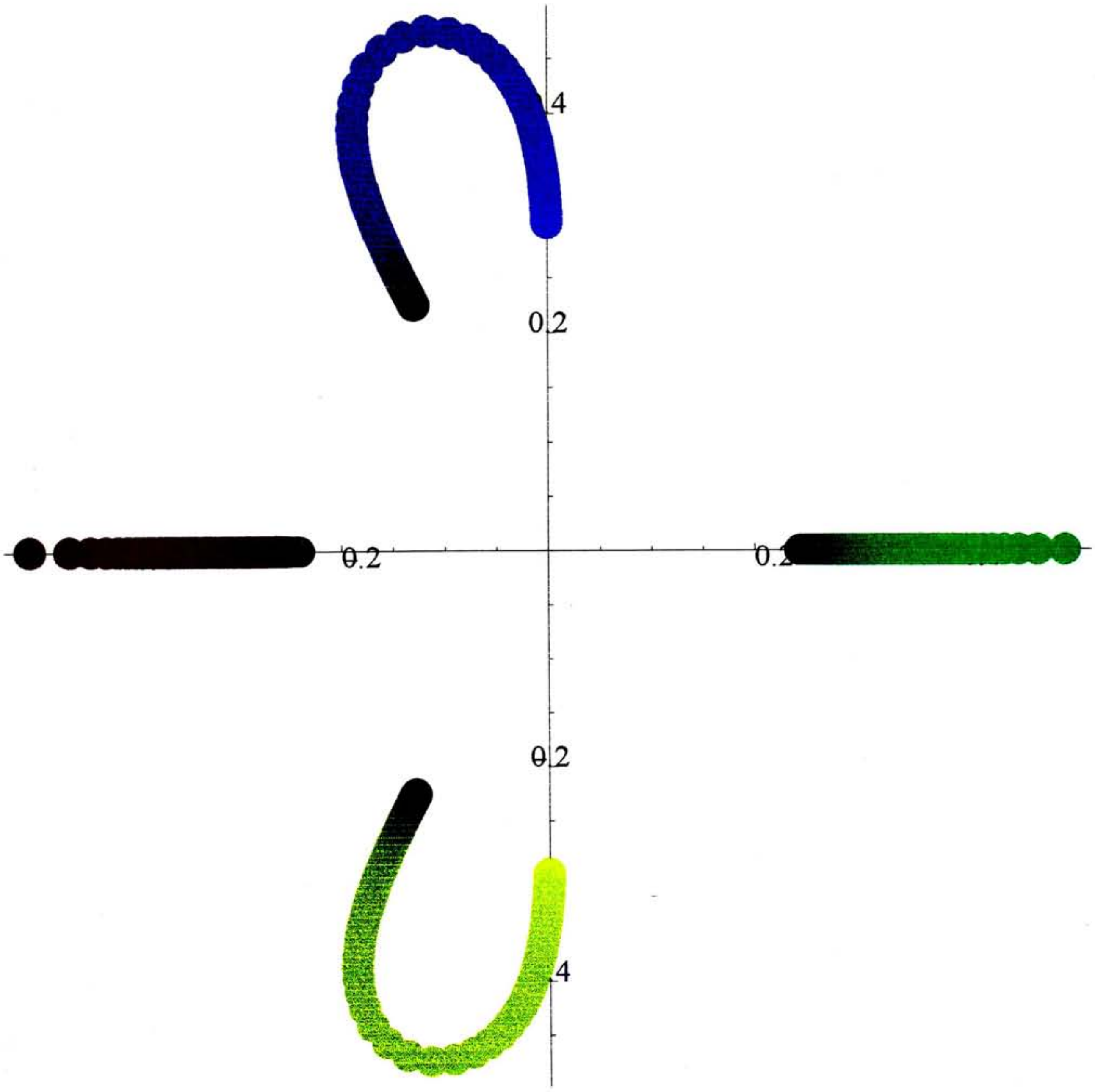


Figure 4.12 The orbits by iterating 0.5, -0.5, 0.3i and -0.3i with

$R(z) = z - 4z^7 + 6z^{10} - 6z^{13} + z^{16}$ for 200 times



Chapter 5 More about Julia Set and Fatou Set

5.1 Some Examples of Julia Set

In the previous chapter, we talk about the definition and some elementary properties of Julia set but we haven't any idea of what a Julia set looks like. So, before the discussion of further properties of the Julia set and Fatou set, we give some simple examples on Julia set first:

EXAMPLE 5.1.1: Let us consider the polynomial, $f(z)=z^2$ as $f^n(z)=z^{2^n}$, therefore, we have $f^n(z)\to 0$ when $|z| < 1$, while $f^n(z)\to\infty$ when $|z| > 1$. The action of the iterates f^n on the unit circle was interesting, by letting $z = e^{i\theta}$, we find that if θ is irrational, the iterates will move round the unit circle in a 'chaotic' fashion (in the sense that it never converge to any point on the unit circle), while if θ is rational, the iterate will finally reach the fixed point and stay there, in fact for this f, the Julia set is the unit circle. Moreover, the forward and the backward iterate of f^n on the unit circle still lying on the circle exhibit an important property of the invariance of the Julia set which will be discussed in the later sections.

EXAMPLE 5.1.2: Consider the rational function $R(z)=\dfrac{(z-2)^2}{z^2}$, the Julia set of this rational function is the whole complex plane and we will come across this example later in our discussion.

EXAMPLE 5.1.3: For the polynomial $P(z) = z^2 - 2$, let's consider the conformal map $h(\zeta) = \zeta + 1/\zeta$ of $\{\,|\zeta| > 1\}$ onto $\mathbf{C}\backslash[-2, 2]$. The identity $P(h(\zeta)) = h^2(\zeta) - 2 = h(\zeta^2)$ gives $h^{-1} \circ P \circ h = \zeta^2$. thus the dynamics of $P(z)$ on $\mathbf{C}\backslash[-2, 2]$ are the same as those of ζ^2 on $\{\,|\zeta| > 1\}$. Since the iterates of any ζ, $|\zeta| > 1$, under ζ^2 tend to ∞, so do the iterates under P of any $z \in \mathbf{C}\backslash[-2, 2]$. Evidently $[-2, 2]$ is invariant under P, so any point in $\mathbf{C}_\infty\backslash[-2, 2]$ tend to ∞ when iterate by P, i.e. the Julia set of P is $[-2, 2]$.

The above examples only show some special examples of Julia set, in fact there are many more examples which are more fascinating and can only be produced graphically by using computers (to accomplish the tedious calculation and plotting). Later in this chapter, we will give illustrations on those beautiful Julia set and show the properties with which the algorithm of producing computer graphics based on.

5.2 Completely Invariant Set

It's interesting to observe that when forward or backward iteration is performed on any point chosen either from the Fatou set or the Julia set. The iteration never brings the chosen point to the other set, therefore, some kind of invariant property seems to exist between these two sets.

DEFINITION 5.2.1: Let g be a map of a set X to itself, a subset U of X is:

(1) forward invariant if $g(U) = U$

(2) backward invariant if $g^{-1}(U) = U$

(3) completely invariant if $g(U) = U = g^{-1}(U)$

PROPOSITION 5.2.2: Let g: $X \to X$ be a surjective map, then backward invariance is equivalent to completely invariant.

PROOF: As g is surjective, ie. $g(X) = X$. It is obvious that completely invariant implies backward invariant. On the other hand, if $U \subseteq X$ is backward invariant ie. $g^{-1}(U) = U$, then $g(g^{-1}(U)) = g(U)$ and by surjectivity, $g(g^{-1}(U)) = U$, therefore $g(U) = U$, thus U is also forward invariant and hence U is completely invariant.

By Proposition 5.2.2, it's important to note that the condition of surjectivity is crucial, in the sense that, there is no difference between backward invariance and complete invariance.

THEOREM 5.2.3: Let R be any rational map, then the Fatou set F is completely invariant under R.

PROOF: Since R is surjective, by proposition 5.2.2, it suffices to prove that F is backward invariant. To prove $R^{-1}(F) \subseteq F$, we take z_0 in $R^{-1}(F)$ and let $\omega_0 = R(z_0)$, ie, $\omega_0 \in F$. By equicontinuity, it follows that for any positive ε, there is a positive δ such that $|\omega - \omega_0| < \delta$, then for all n, $|R^n(\omega) - R^n(\omega_0)| < \varepsilon$. As R is continuous, there is also a positive ρ such that if $|z - z_0| < \rho$, then $|R(z) - R(z_0)| < \delta$, and hence $|R^{n+1}(z) - R^{n+1}(z_0)| < \varepsilon$. This shows that $\{R^{n+1}: n \geq 1\}$ is equicontinuous at z_0. Thus $\{R^n: n \geq 1\}$ is equicontinuous at z_0 and hence on F. As $R^{-1}(F)$ is open, we deduce that $R^{-1}(F) \subseteq F$. To prove the opposite inclusion, take any z_0 in F and let $\omega_0 = R(z_0)$. Since

$z_0 \in F$, give any positive ε, there is a positive δ such that for all n (by the condition of equicontinuity) if $|z - z_0| < \delta$, then $|R^{n+1}(z) - R^{n+1}(z_0)| < \varepsilon$. The set of z satisfying $|z - z_0| < \delta$, is an open neighbourhood N, of z_0, and so $R(N)$ is an open neighbourhood of ω_0. If $\omega \in R(N)$, then there is an $z \in N$ such that $\omega = R(z)$. therefore

$$|R^n(\omega) - R^n(\omega_0)| = |R^{n+1}(z) - R^{n+1}(z_0)| < \varepsilon$$

that is $\omega_0 \in F$, therefore $F \subseteq R^{-1}(F)$ and hence $F = R^{-1}(F)$and that F is a completely invariant set.

COROLLARY 5.2.4: Julia set, J, is a completely invariant set.

PROOF: $J = \mathbf{C}_\infty \backslash F$, it suffices to prove $J = R^{-1}(J)$

Take $y \in R^{-1}(\mathbf{C}_\infty \backslash F)$

$\Rightarrow$ $R(y) \in \mathbf{C}_\infty \backslash F$ $\Rightarrow$ $R(y) \notin F$

$\Rightarrow$ $y \notin F$ (Since F is completely invariant, if $y \in F \Rightarrow R(y) \in F$ which is a contradiction)

$\Rightarrow$ $y \in \mathbf{C}_\infty \backslash F$ (that is $y \in J$)

Take $y \in \mathbf{C}_\infty \backslash F$

$\Rightarrow$ $y \notin F$

$\Rightarrow$ $y \notin R^{-1}(F)$ (Since F is completely invariant)

$\Rightarrow$ $y \in R^{-1}(\mathbf{C}_\infty \backslash F)$ (that is $y \in R^{-1}(J)$)

By combining the results, $R(J) = J$ which proves that J is backward invariant and thus completely invariant.

5.3 Exceptional Sets

DEFINITION 5.3.1: Let R be a rational map of degree more than one, $E(R)$ is the exceptional set if it's a finite and completely invariant set, the elements in the exceptional set are called exceptional points.

THEOREM 5.3.2: $E(R)$ has at most two elements.

PROOF: Suppose $E(R)$ has k elements $\{\omega_1, \ldots, \omega_k\}$, then R acts as a permutation for elements in $E(R)$, since $E(R)$ has finite number of elements, therefore there is a p such that $R^p(\omega_i) = \omega_i$ for $i = 1, 2, \ldots, k$, suppose $\deg(R^p) = d$, then there are d solutions for $R^p(z) = \omega_i$ in which all solutions are ω_i's, therefore by counting multiplicity, there are $k(d-1)$ critical points for R^p. By Riemann Hurwitz Formula

$$k(d-1) \leq 2(d-1) \text{ that is } k \leq 2$$

and that $E(R)$ has at most 2 elements.

THEOREM 5.3.3: Let R be a rational map with $\deg(R) \geq 2$, then the exceptional points of R lie in $F(R)$.

PROOF: From Theorem 4.3.2, $E(R)$ has at most 2 elements, therefore there are 3 cases:

(1) $E(R) = \varnothing$, there's nothing more to discuss.

(2) For $E(R)$ to have one element, say $\{\zeta\}$, after a suitable conjugation by Mobius transformation, $g = \dfrac{1}{z-\zeta}$. Let $R' = g^{-1}Rg$, then $E(R') = \{\infty\}$, since $E(R')$ is an invariant set, therefore $R'^{-1}(\infty) = \{\infty\}$, ie. R' is a polynomial. Next, we are going to prove that ∞ is in

$F(R')$. It is obvious that R'^n converges uniformly to ∞ on some neighbourhood W of ∞, therefore, given any positive ε and any z in W, there is a positive integer N such that, if $n \geq N$, we have $\sigma(R'^n(z), \infty) < \varepsilon/2$ (where σ is the chordal metric in $\mathbf{C}_\infty$). Thus if z, ω are in W, then $\sigma(R'^n(z), R'^n(\omega)) \leq \sigma(R'^n(z), \infty) + \sigma(\infty, R'^n(\omega)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, so $\{R'^n\}$ is equicontinuous in W and ∞ is in $F(R')$. By theorem 4.2.3, ζ is in $F(R)$.

(3) For $E(R)$ to have 2 elements, $\{\zeta_1, \zeta_2\}$, we have 2 cases:

(a) $R(\zeta_1) = \zeta_1$ and $R(\zeta_2) = \zeta_2$, by a suitable conjugation, we have $R'(0) = 0$ and $R'(\infty) = \infty$, where $R' = g^{-1}Rg$ and $g = \dfrac{z-\zeta_1}{z-\zeta_2}$, it should be noted from above that R' is a polynomial with also the condition that $R'^{-1}(0) = \{0\}$, then R' should be in the form az^d from some positive integer d, with a similar argument as above $\{\zeta_1, \zeta_2\} \in F(R)$.

(b) $R(\zeta_1) = \zeta_2$ and $R(\zeta_2) = \zeta_1$, by the same conjugation used in 3(a), we have $R'(0) = \infty$ and $R'(\infty) = 0$ and R' has all of its zeros and poles in $\{0, \infty\}$, so it is of the form az^d where d is a negative integer, it should be noted that R'^2 fixes ∞ and 0, therefore R'^2 is a polynomial of the form $a'z^{d^2}$, by the same argument as 3a: $\{\infty, 0\} \in F(R'^2)$, by theorem 4.2.4, $F(R'^2) = F(R')$ that is $\{\infty, 0\} \in F(R')$ and completes the proof.

REMARKS: For Mobius transformation, the exceptional set is the set containing the fixed point (since the pullback of any point contains one element only), therefore, from the discussion of section 4.3, we have a non-empty exceptional set with at most 2 elements. Also, from the above argument, we see that for rational map R to have exceptional points, R and R^2 are conjugate to a polynomial, therefore it is obvious that most rational maps have no exceptional points.

5.4 Properties of Julia Set

To begin our discussion on Julia Set, it is natural to ask whether the Julia set is empty or not, we answer the question by the following theorem:

THEOREM 5.4.1: If $\deg(R) \geq 2$, then $J(R)$ is non empty.

PROOF: Suppose that J is empty, then the family $\{R^n\}$ is normal on the entire complex sphere, that is, for any point z, we could find a neighbourhood W and a subsequence R^{n_j}, which would converge to an analytic function f: $W \to \mathbf{C}_\infty$, since R^{n_j} converges uniformly on the complex sphere to f then f is a rational function. Let

$\deg(f) = d < \infty$. As f is the limit of function whose degree tend to ∞, it must have degree ∞, contradicts that $\deg(f) < \infty$, therefore $J(R)$ is non-empty.

By Corollary 5.2.4, we know that $J(R)$ is completely invariant and by above, we have also that $J(R)$ is non-empty, that $J(R)$ may be infinite or finite, but as we know that a finite invariant set is an exceptional set which is a subset of $F(R)$, thus it constitute the proof of the following corollary:

COROLLARY 5.4.2: **If $\deg(R) \geq 2$, then $J(R)$ is infinite.**

Let R be a rational map with $\deg(R) \geq 2$, in order to draw more information about the structure of the Julia set, it is important to have more information about any closed completely invariant subset, E, of the complex sphere and the Julia set. If E is finite, then it must be a subset of the exceptional set, $E(R)$ which belongs to $F(R)$. If E is infinite, then we have:

THEOREM 5.4.3: $J(R) \subseteq E$

Proof: As E is completely invariant, so is its complement, Ω; hence each R^n maps the open set Ω into itself. Since E is infinite, therefore, we can apply Montel's theorem to the family $\{R^n\}$ on Ω, this shows that $\{R^n\}$ is normal in Ω, and therefore $\Omega \subseteq F(R)$ that is $J(R) \subseteq E$.

REMARK: we can conclude from the above that $J(R)$ is the minimal, closed and completely invariant set with infinite number of elements. This, in some sense, agree with the definition of Fatou set which states that $F(R)$ is the maximal open set on which $\{R^n\}$ is equicontinuous and $J(R)$ is the complement of $F(R)$.

In 5.1, we have some examples of Julia set which are smooth curves, but in general, it is not the case, the following theorems will give us a general picture of what a Julia set looks like:

THEOREM 5.4.4: Either $J(R) = \mathbf{C}_\infty$ or $J(R)$ has empty interior where $\deg(R) \geq 2$.

Proof: Suppose $J \neq \mathbf{C}_\infty$, then J consists of its interior, $J^{\,o}$ and its boundary, ∂J, therefore, $\mathbf{C}_\infty = \partial J \cup J^{\,o} \cup F$. F and J are completely invariant and so are ∂J and $J^{\,o}$. The completely invariance of $J^{\,o}$ can be seen from the argument that R is continuous on $\mathbf{C}_\infty$, $R^{-1}(J^{\,o})$ is an open subset of $R^{-1}(J) = J$, therefore, $R^{-1}(J^{\,o}) \subseteq J^{\,o}$. Similarly, as R is an open map, $R(J^{\,o})$ is an open subset of J and so $R(J^{\,o}) \subseteq J^{\,o}$. Thus $J^{\,o} \subseteq R^{-1}(R(J^{\,o})) \subseteq R^{-1}(J^{\,o})$ and so $J^{\,o}$ is completely invariant. As $\partial J = J \setminus J^{\,o}$, and apply the same argument as corollary 4.2.4, we can show that ∂J is also completely invariant. Since $J \neq \mathbf{C}_\infty$, therefore F is non-empty, then $F \cup \partial J$ is an infinite, closed, completely invariant set, by the minimality of J, we have $J \subseteq F \cup \partial J$, ie. $J \subseteq \partial J$, since $J \cap F = \varnothing$ and hence $J = \partial J$, that means J has empty interior.

THEOREM 5.4.5 J has no isolated points.

Proof Let J' be the derived set of J, it suffices to show that $J' = J$, since J is closed, therefore $J' \subseteq J$. To prove the opposite inclusion: As J is infinite, therefore J' is non empty, also J' is closed, since it's a derived set and R is continuous, it is clear that $R(J') \subseteq J'$, hence $J' \subseteq R^{-1}(J')$. In addition, R is an open map, it is not hard to see that $R^{-1}(J') \subseteq J'$, and hence J' is completely invariant, J' cannot be finite, since then $J' \subseteq E(R) \subseteq F(R)$, that is J' is infinite and hence J' is an infinite, closed, completely invariant set and by the minimality of J, $J \subseteq J'$ and hence $J = J'$.

Now, we can draw some conclusions to the general shape of a Julia set, firstly it's 'thin' and secondly, Julia set appears as a 'whole body', that is, it does not have any detached point from the main 'body'. This agrees with the observation in the examples given in 4.1.

5.5 Forward and Backward Convergence of Sets

This section begins by defining the *backward orbit* of any point z which is the set

$$O^{-}(z) = \{\omega : \text{for some } n \geq 0,\ R^{n}(\omega) = z\} = \bigcup_{n\geq 0} R^{-n}(z)$$

and the points in $O^{-}(z)$ are called the *predecessors* of z.

In the last chapter, we discuss about the iteration of points and study the orbits of points and hence the convergence(*stability*) of those points. Now we are going to focus our discussion on the iteration of sets, as we know that rational functions are analytic on $\mathbf{C}_\infty$, therefore, by open mapping theorem, it maps open sets to open sets, hence, we can study the convergence of a set by iterating it successively by R. The following discussion is important for it's a direct consequence of the Montel's Theorem. In order to have a better understanding of the theorem, a simple example will come before the proof which serves as an illustration of the theorem.

THEOREM 5.5.1: Let R be a rational map of degree greater than one and let W be any non-empty open set which meets J. Then

(i) $\bigcup_{n=0}^{\infty} R^{n}(W) \supset \mathbf{C}_\infty - E(R)$; and

(ii) for all sufficiently large integer, n, $J \subset R^n(W)$

EXAMPLE 5.5.2: In example 5.1.1, we know that the Julia set of the rational function $R(z) = z^2$ is the unit circle with O as its centre. This quadratic function has been chosen for our illustration due to its relatively simpler nature of Julia set. Now we take W as the open disk of radius 0.2 unit and with centre at 1. It is clear that W meets J. The region bounded by the closed curve in red (with the boundary of W which is in blue and serve as a reference) presented from Figure 5.1 to Figure 5.4 show the set, $R^n(W)$ for different values of n (from $n = 1$ to $n = 4$). We noted that the region is expanding as n increases from 1 to 3, when n is greater than 3, the region still expanding (in terms of size) but it's interesting to note that the region does not contain the origin and note the special feature of the region near the origin for it's a critical point there. As we consider the union of the region presented in the above figures, we can see from figure 5.5 to 5.8, $\bigcup_{k=0}^{n} R^k(W)$ is expanding as value of n increases and this tendency tells us that as $k \to \infty$, $\bigcup_{k=0}^{\infty} R^k(W)$ covers almost the whole plane. Also, when n is sufficiently large, the unit circle is covered by $R^n(W)$.

Figure 5.1 The image of $R^n(\overline{W})$ when n =1

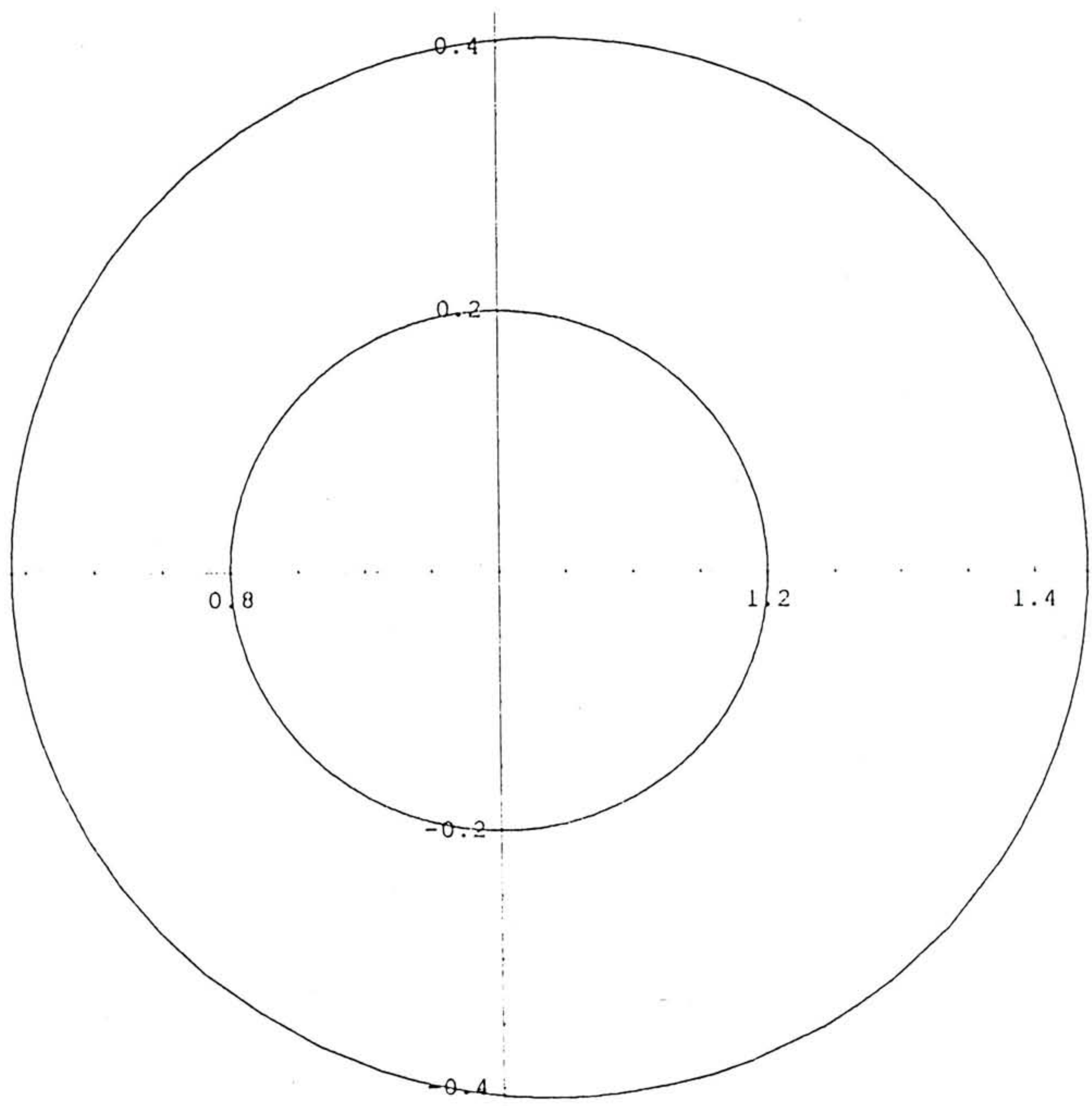


Figure 5.2 The image of $R^n(W)$ when $n = 2$

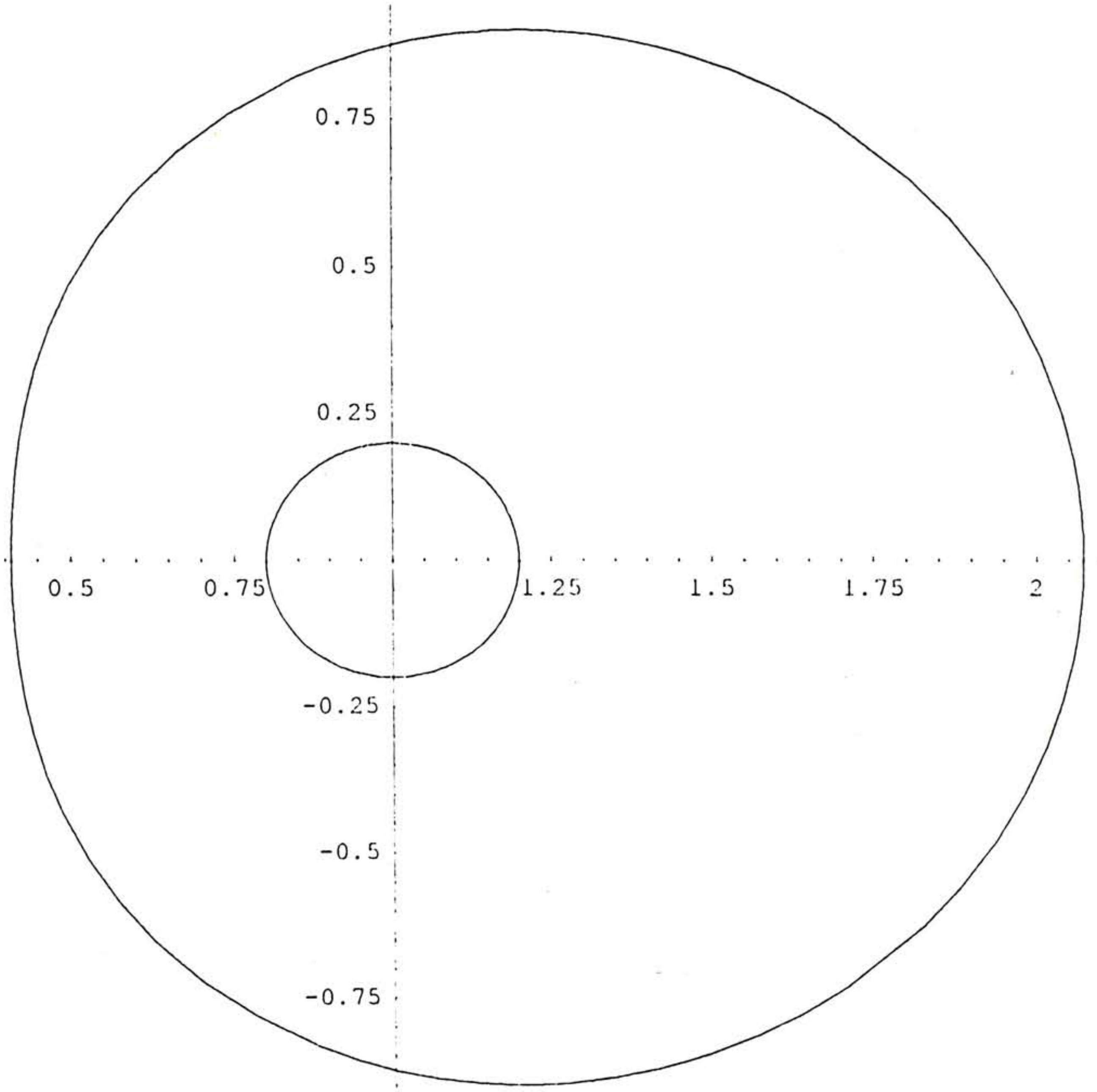


Figure 5.3 The image of $R^n(W)$ when $n = 3$

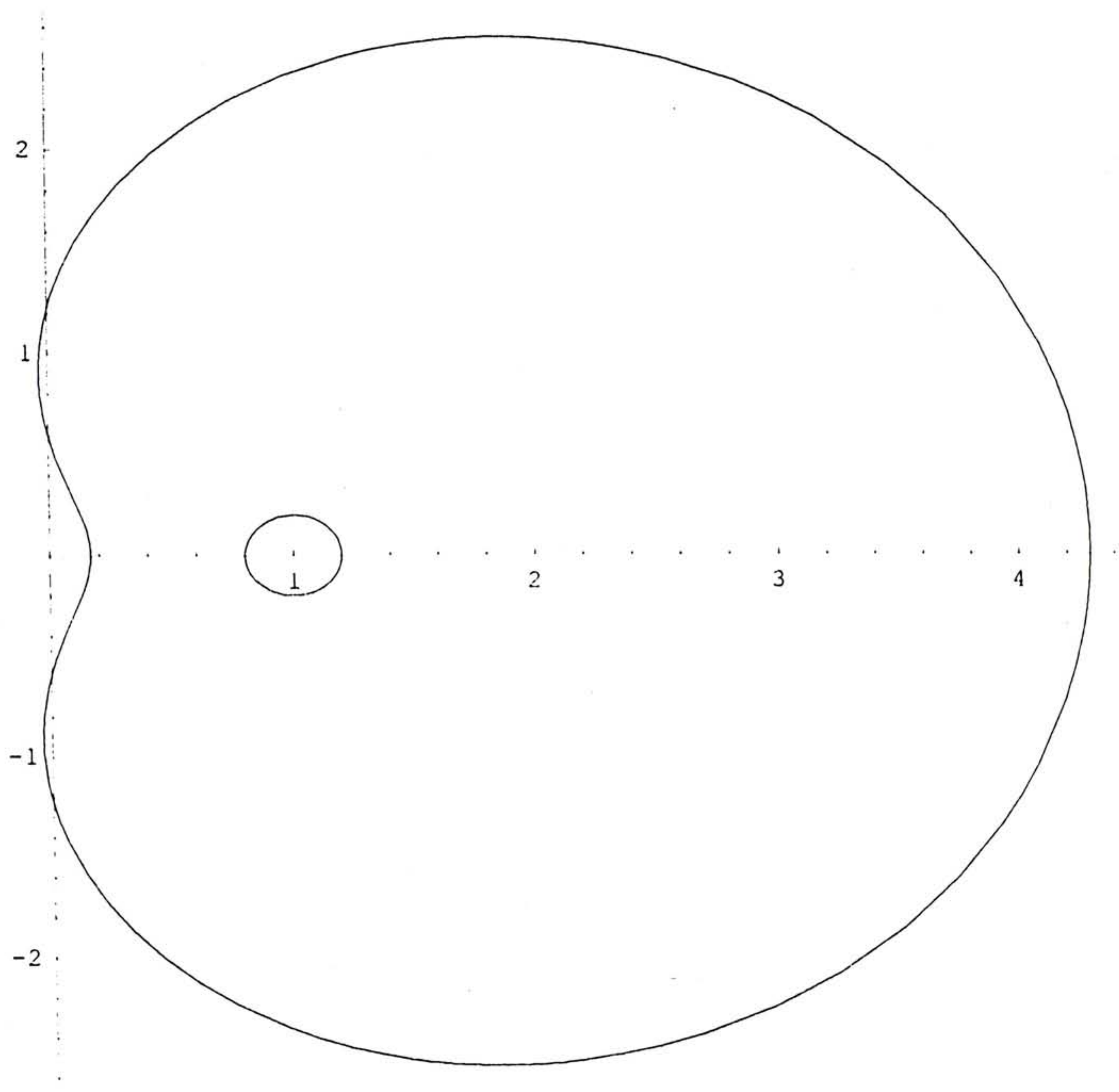


Figure 5.4 The image of $R^n(W)$ when $n = 4$

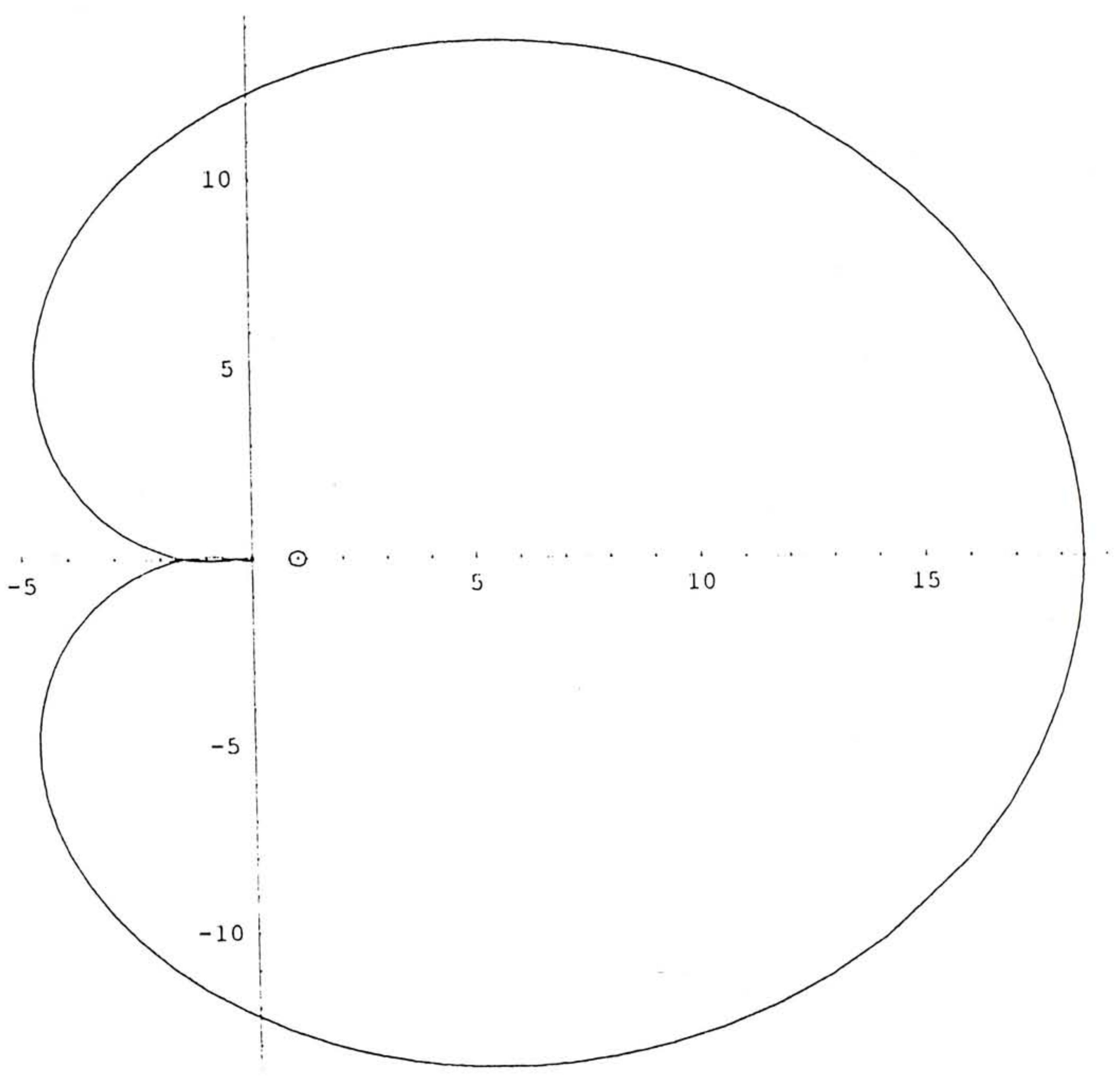


Figure 5.5 The union of the images of $R(W)$ and $R^2(W)$

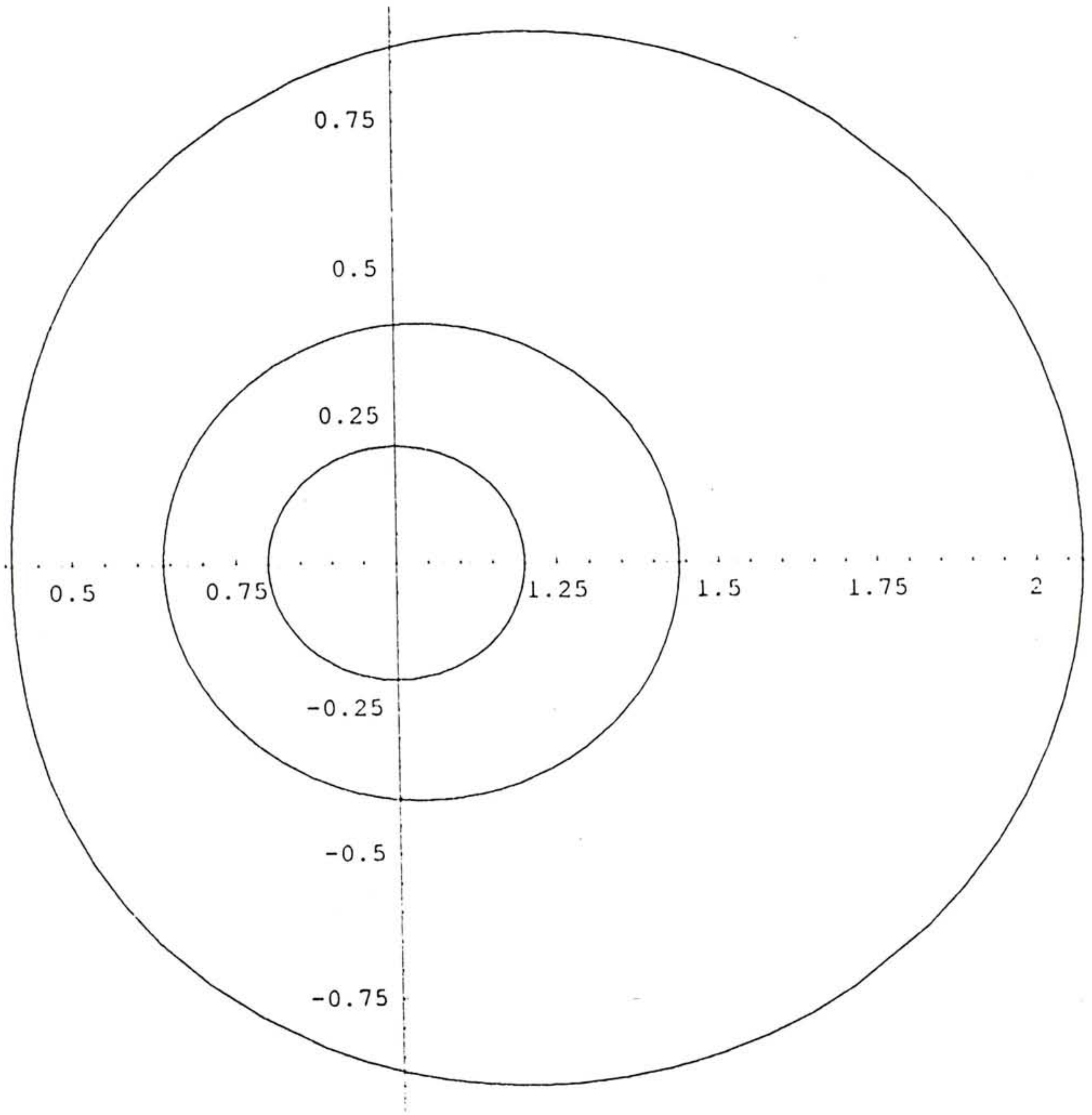


Figure 5.6 The union of images of $R(W)$, $R^2(W)$ and $R^3(W)$.

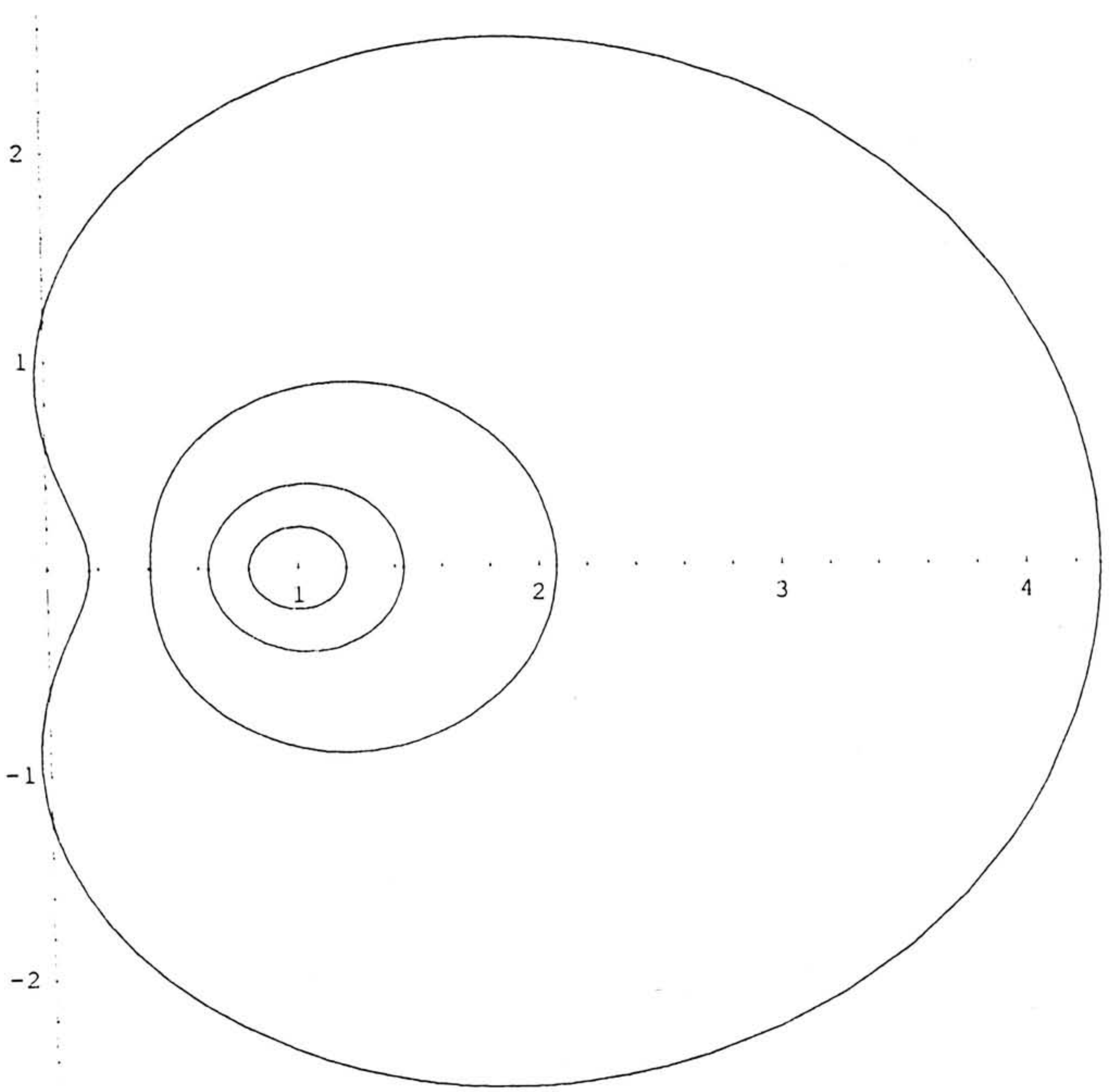


Figure 5.7 The union of images of $R(W)$, $R^2(W)$, $R^3(W)$ and $R^4(W)$.

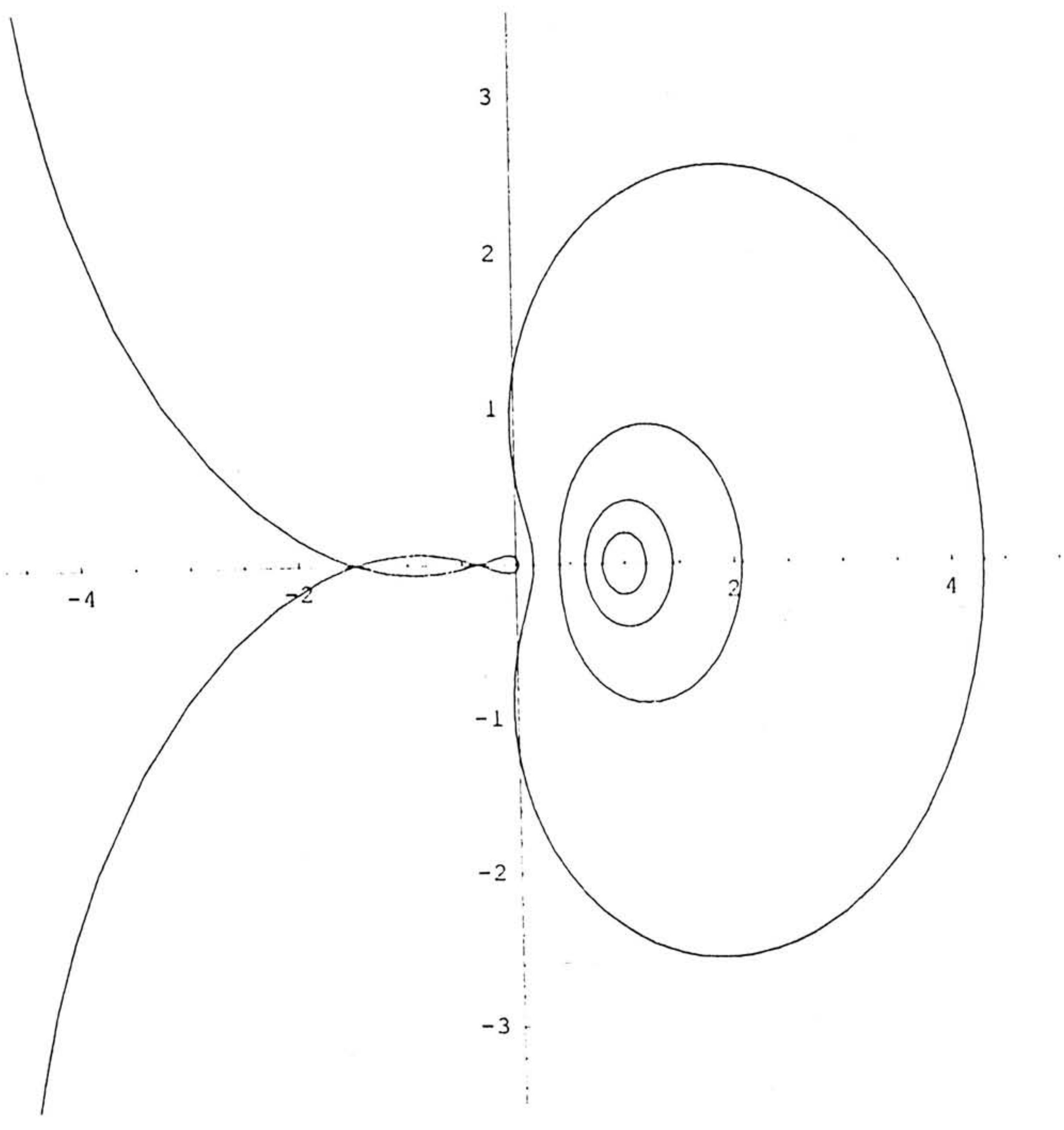
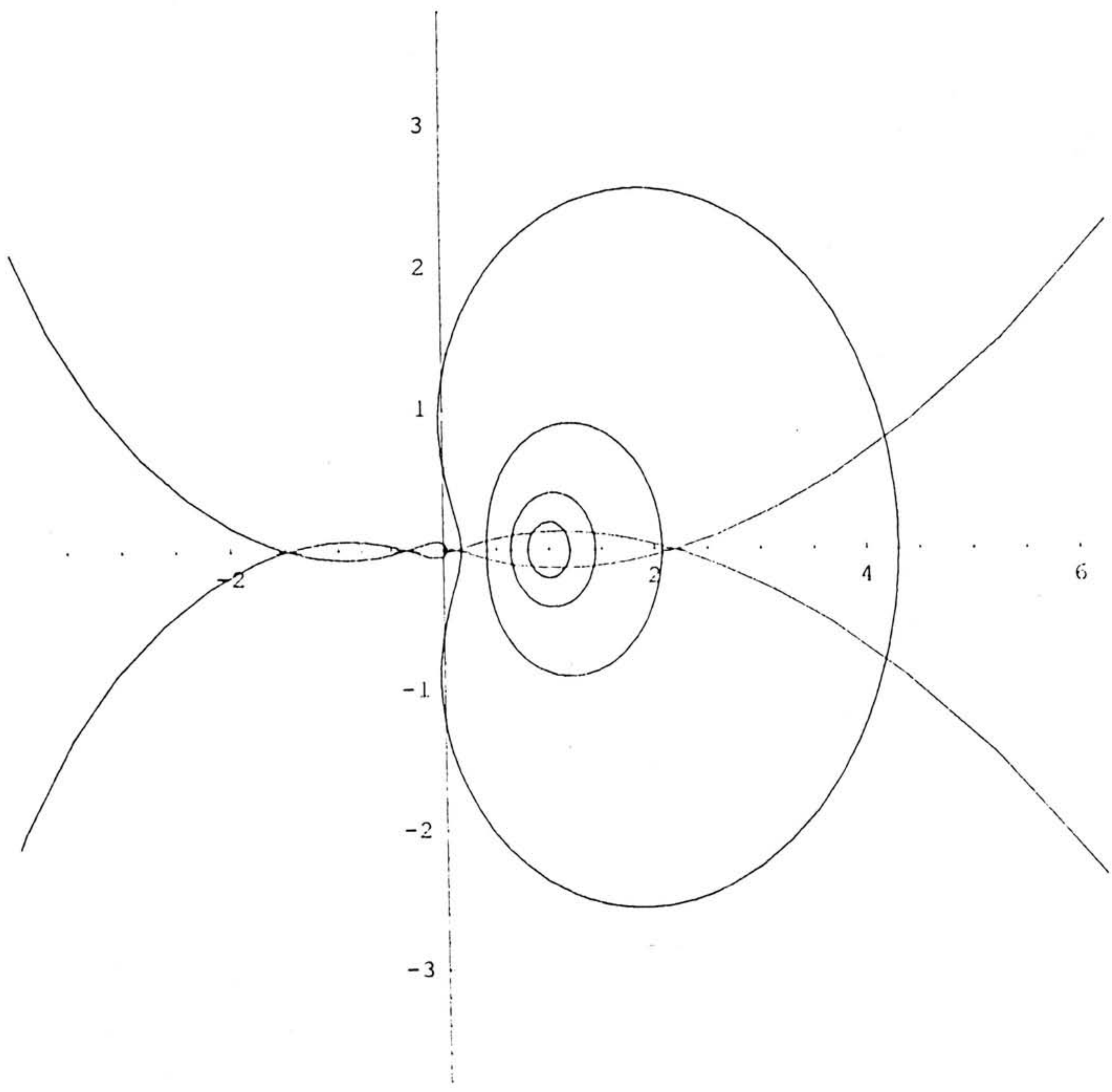


Figure 5.8 The union of images of $R(W)$, $R^2(W)$, $R^3(W)$, $R^4(W)$ and $R^5(W)$.



Proof of Theorem 5.5.1:

(i) $\mathbf{C}_\infty \setminus \bigcup_{k=0}^{\infty} R^k(W)$ must contain at most 2 elements, otherwise, by Montel theorem $\{R^n\}$ is normal in W, i.e. $W \subset F$, which is a contradiction. Let z is not exceptional, then $z \in \mathbf{C}_\infty \setminus E(R)$, then z has an infinite backward orbit and must meet $\bigcup_{k=0}^{\infty} R^k(W)$, then for some point ω, there are non-negative integers p and q, such that $R^p(\omega) = z$ and $\omega \in R^q(W)$, that is $z \in R^{p+q}(W) \subset \bigcup_{k=0}^{\infty} R^k(W)$.

(ii) In W, choose 3 open sets, W_1, W_2 and W_3, each meeting J and a positive chordal distance apart from each other. By applying Montel's theorem, we have $W_k \subset R^n(W_j)$, putting $R^n = S$, we have $W_j \subset S(W_j)$, therefore constructing a sequence of sets $\{S^m(W_j)\}$ which is increasing. Applying (i) to S and W_j, then, there exists n, such that $J \subseteq R^n(W_j) \subset R^n(W)$, since J is invariant, therefore, $J = R(J) \subseteq R(R^n(W)) = R^{n+1}(W)$ and (ii) follows by induction.

The following two results tells us what J actually is: a closure of some sets. The first result is important since it's the theoretical basis of the one of the algorithm for plotting J by computers.

THEOREM 5.5.3: Let R be a rational map with $\deg(R) > 1$.

(i) If z is not exceptional, then J is contained in the closure of $O^-(z)$.

(ii) If $z \in J$, then J is the closure of $O^{-}(z)$.

By the above theorem, we can choose a non-exceptional point and then computes successive backward images of it to plot the set J, this is the basic idea of one of the algorithm to illustrate J which is called the *backward iterate method.* For this method, we have difficulties in handling the rapid increase in the number of the inverse images causing inconvenient in computing inverse image in the following stages, but, this can be overcome by making choices of the branch.

EXAMPLE 5.5.4: Figure 5.9 to 5.14 show the plot of Julia sets of various rational functions by applying the backward iterate method.(For computer programs written in Mathematica to illustrate Julia set, see the appendix).

Remarks: It's not easy to find the inverse of image of a rational function of degree greater than 2, therefore most of the examples shown are plotting of Julia set of the rational function with degree 2.

From the figures shown, we can observe that the Julia set, in general, may not be smooth, as those shown in example 5.1.1 and the Fatou set may not be simply connected.

Proof of theorem 5.5.3: Consider any non-exceptional point z and any non-empty open set W which meets J. As W meets J, theorem 5.5.3 implies that z lies in some $R^n(W)$ and so $O^{-}(z)$ meets W and the results of (i) follows.

If $z \in J$, then the closure of $O^{-}(z) \subseteq J$, together with (i), we have (ii).

Figure 5.9 The Julia set of $R(z) = z^2$

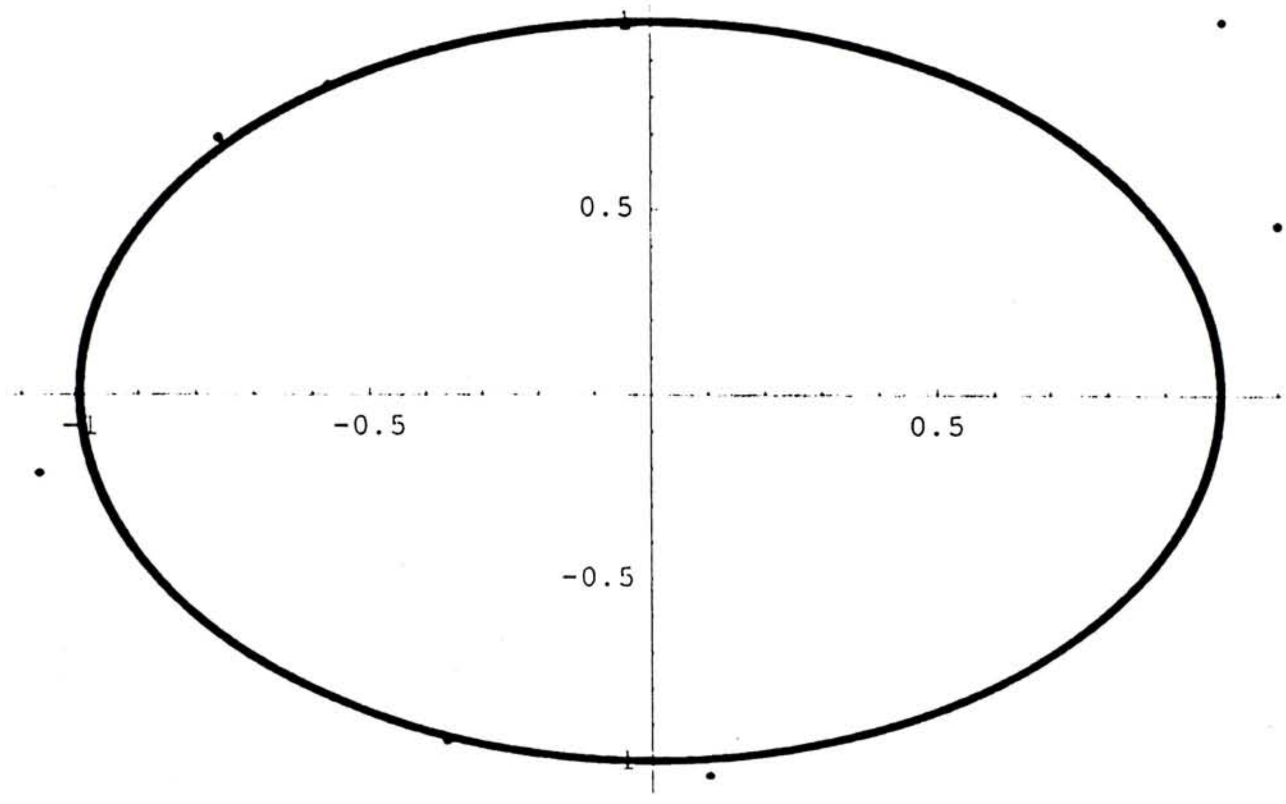


Figure 5.10 The Julia set of $R(z) = z^2 - 2$

(Note from the figure that the Julia set is not a line segement as in our dicussion in example 5.1.3, and we can observe that the points accumulate on the line segment within the range of -2 and 2, since many points lie within a narrow range of the imaginary axis. This can be explained by the truncation in the calculation performed by the computers.)

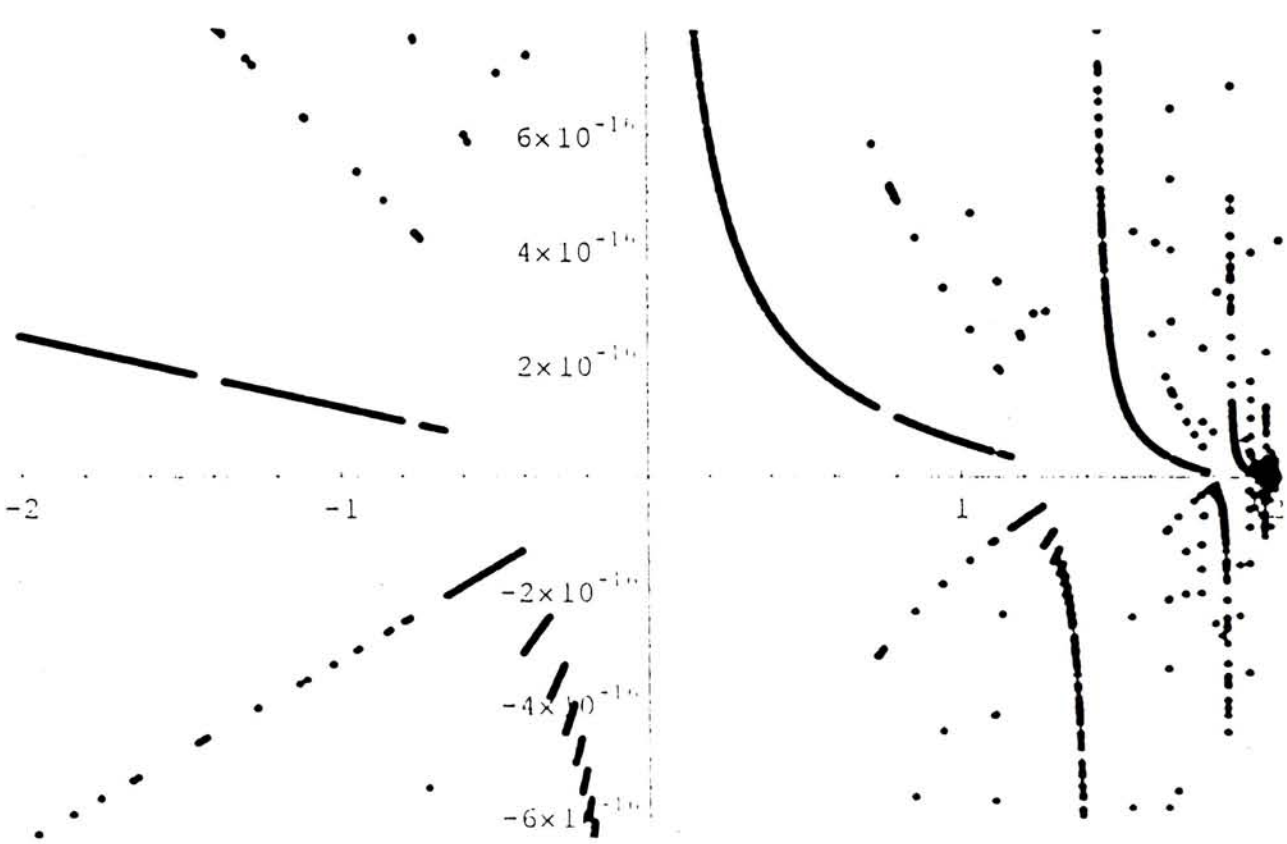


Figure 5.11 The Julia set of $R(z) = z^2 + 1$

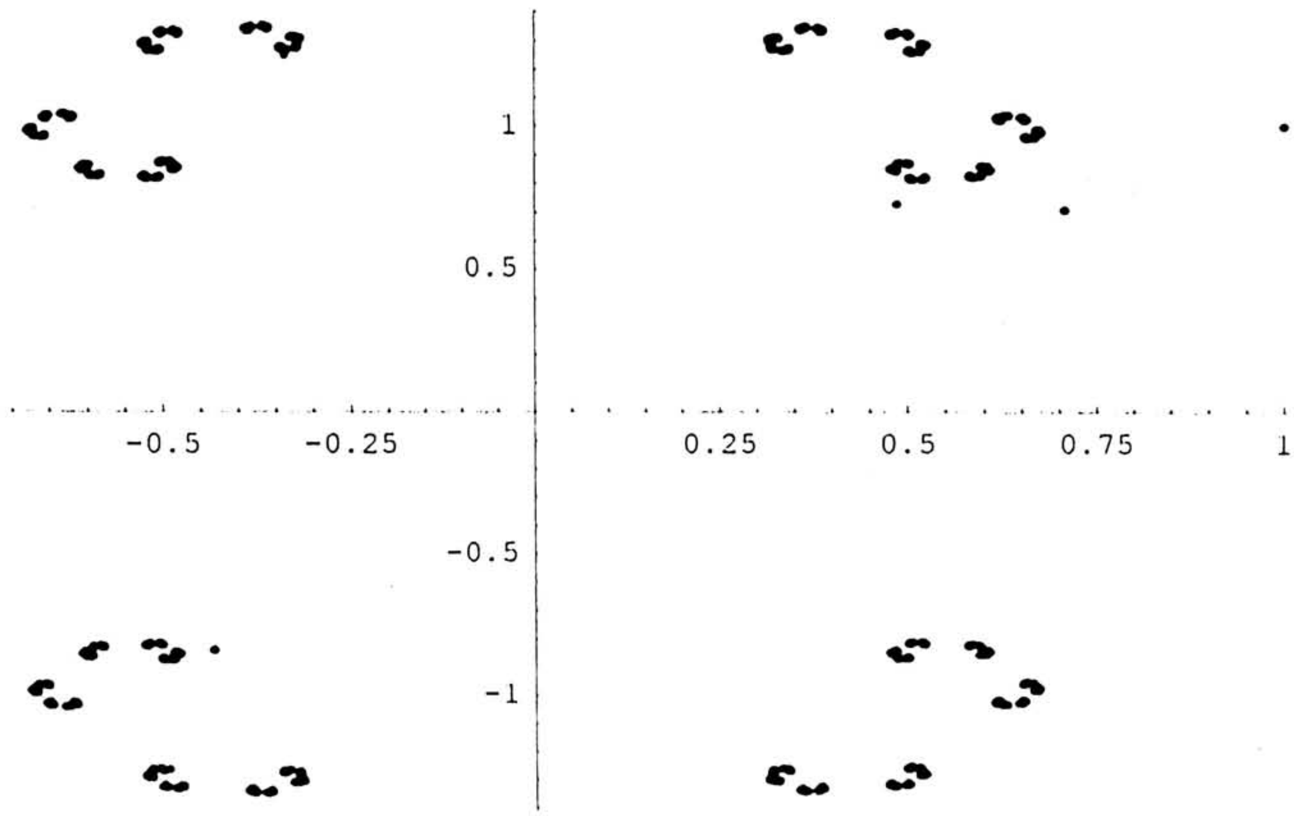


Figure 5.12 The Julia set of $R(z) = z^2 - i$

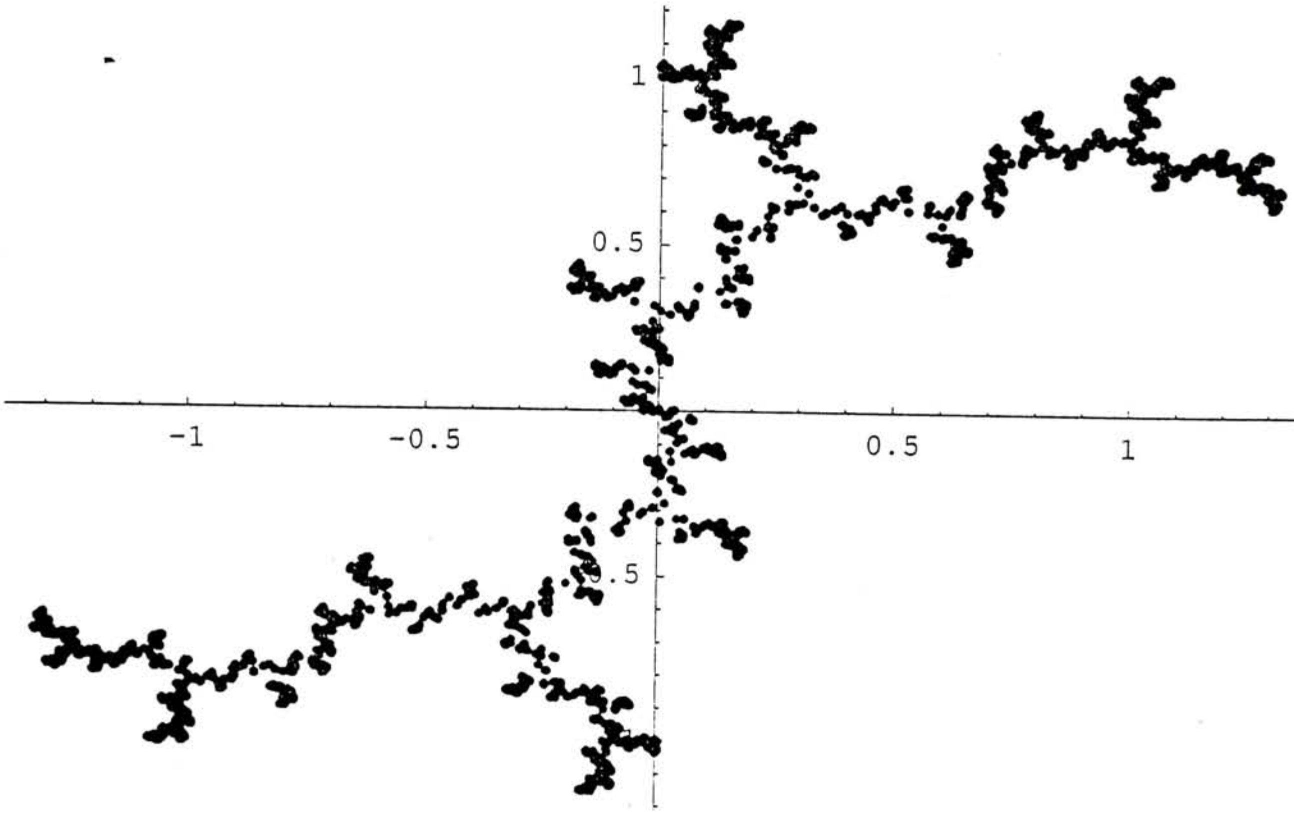


Figure 5.13 The Julia set of $R(z) = z^2 - 1$

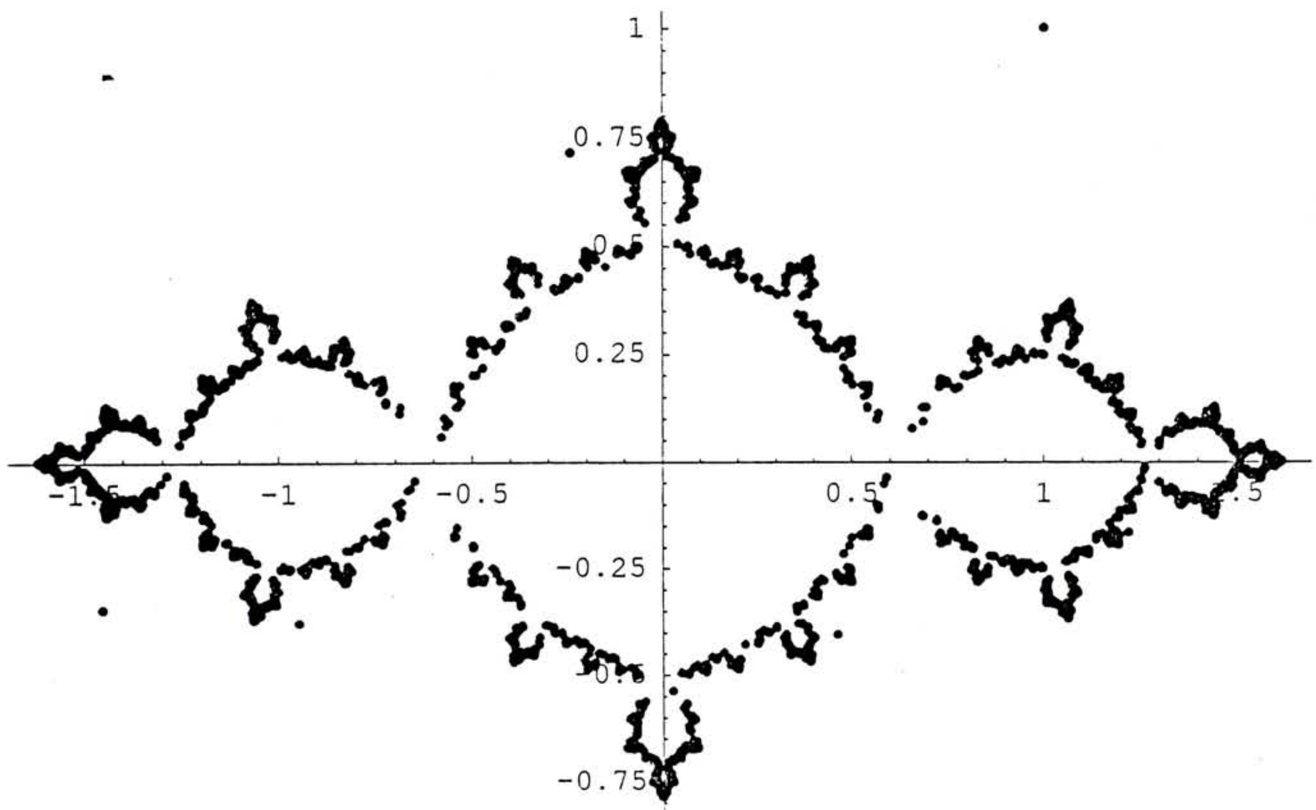
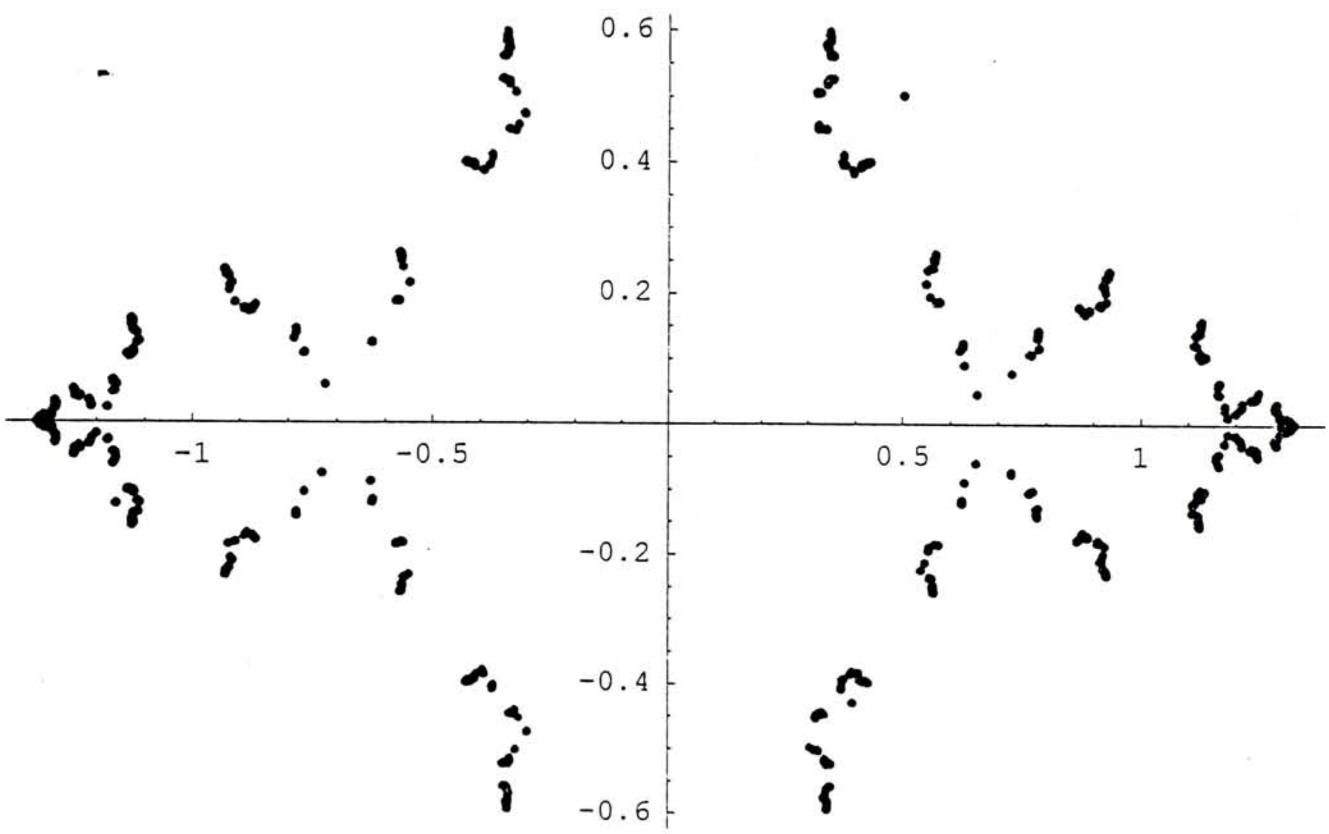


Figure 5.14 The Julia set of $R(z) = z^3 - 1$



THEOREM 5.5.5: Suppose that $\deg(R) > 1$, then J is contained in the closure of the set of periodic points of R.

The theorem implies that there are infinitely many periodic points and also not all periodic points are in J, in fact, the periodic points so involved are repelling. This theorem suggest only the composition of J but not so useful in the construction of the Julia set.

Proof: Let N be any open set meeting J, choose $\omega \in J \cap N$, which is not a critical value of R^2, then $R^{-2}\{\omega\}$ have at least 4 distinct points, choose ω_1, ω_2 and ω_3 distinct from ω, construct neighbourhood N_0, N_1, N_2, N_3 with pairwise disjoint closures which contain ω, ω_1, ω_2 and ω_3 respectively, then R^2 is a homeomorphism from N_j to N_0, where $j = 1$, 2 and 3. Let S_j be the inverse of R^2, then by applying Montel's theorem, there is some $z \in N_0$, some j in $\{1, 2, 3\}$ and some $n \geq 1$ such that $R^n(z) = S_j(z)$ which implies that $R^{2+n}(z) = R^2S_j(z) = z$ and so z is a periodic point in N.

In Theorem 5.5.3, we talk about approximating the Julia set by plotting the closure of the backward orbit. The next theorem, suggest a method of approximating the Julia set by backward iteration of some chosen sets.

THEOREM 5.5.6: Let R be a rational map of degree greater than one and let E be a compact subset of the complex sphere with the property that for all z in $F(R)$, the sequence

$\{R^n(z): n \geq 1\}$ does not accumulate at any point of E. Then given any open set U which contains $J(R)$, $R^{-n}(E) \subseteq U$ for all sufficiently large n.

The theorem tells us that if E is properly chosen, $R^{-n}(E)$ converges to J. E can easily be found by the following method: Let F be a component of the Fatou set which contains an attracting fixed point, ζ. E is any compact subset of $F \setminus \{\zeta\}$. We are going to illustrate the theorem by the following example.

EXAMPLE 5.5.7: Let R be z^2, take E be the disc with 1/2 as its centre with radius 1/4 unit. Figure 5.15 to 5.19 show the action of R^{-n} on E from $n = 1$ to 5. We can see that the iterated set becomes smaller and approaching the unit circle (the Julia set) as n increases.

Proof of theorem 5.5.6 We suppose that the conclusion is false: then there exists some open neighbourhood U of J, and, for n in some sequence $\{n_1, n_2, \ldots\}$, points z_n in $R^{-n}(E)$, but not in U. Now without loss of generality, the points z_n converge to ω, say, and as U is open, ω is not in U. But $J \subset U$; hence ω is in F. Now take any positive ε. As $\omega \in F$, $\{R^n\}$ is equicontinuous in some neighbourhood of ω, so there is a positive δ such that for all n, $\sigma(z, \omega) < \delta$ implies that $\sigma(R^n(z), R^n(\omega)) < \varepsilon$ and as $R^n(z_n) \in E$, this shows that the sequence $R^n(\omega)$ accumulates at E, contrary to our assumption. The proof is completed.

REMARKS: There is another algorithm for computing J of a polynomial, P, the *Boundary Scanning Method*:

We defined the union of the Julia set J and the bounded components of F as the *filled-in Julia set* of P, denoted by $\mathfrak{K}$. Therefore $z \in \mathfrak{K}$ if and only if the iterates $P^n(z)$ are bounded. This property can be taken as a basis for drawing computer pictures of $\mathfrak{K}$ and of J.

Figure 5.20 to 5.23 show respectively the filled-in Julia set of $f(z) = z + z^4$,

$f(z) = \dfrac{3\sqrt{3}}{2} z(z+1)(z+2)$, $f(z) = z - z^4$ and $f(z) = z^2 - 1$.

Figure 5.15 The Image of $R^{-1}(E)$

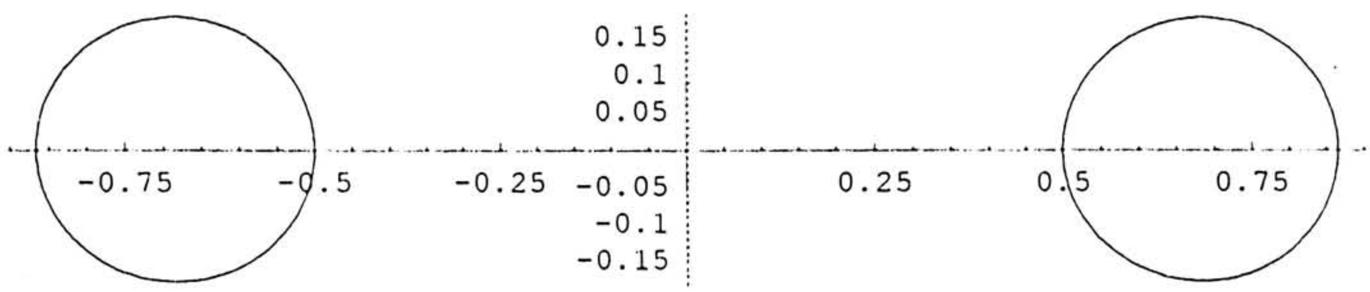
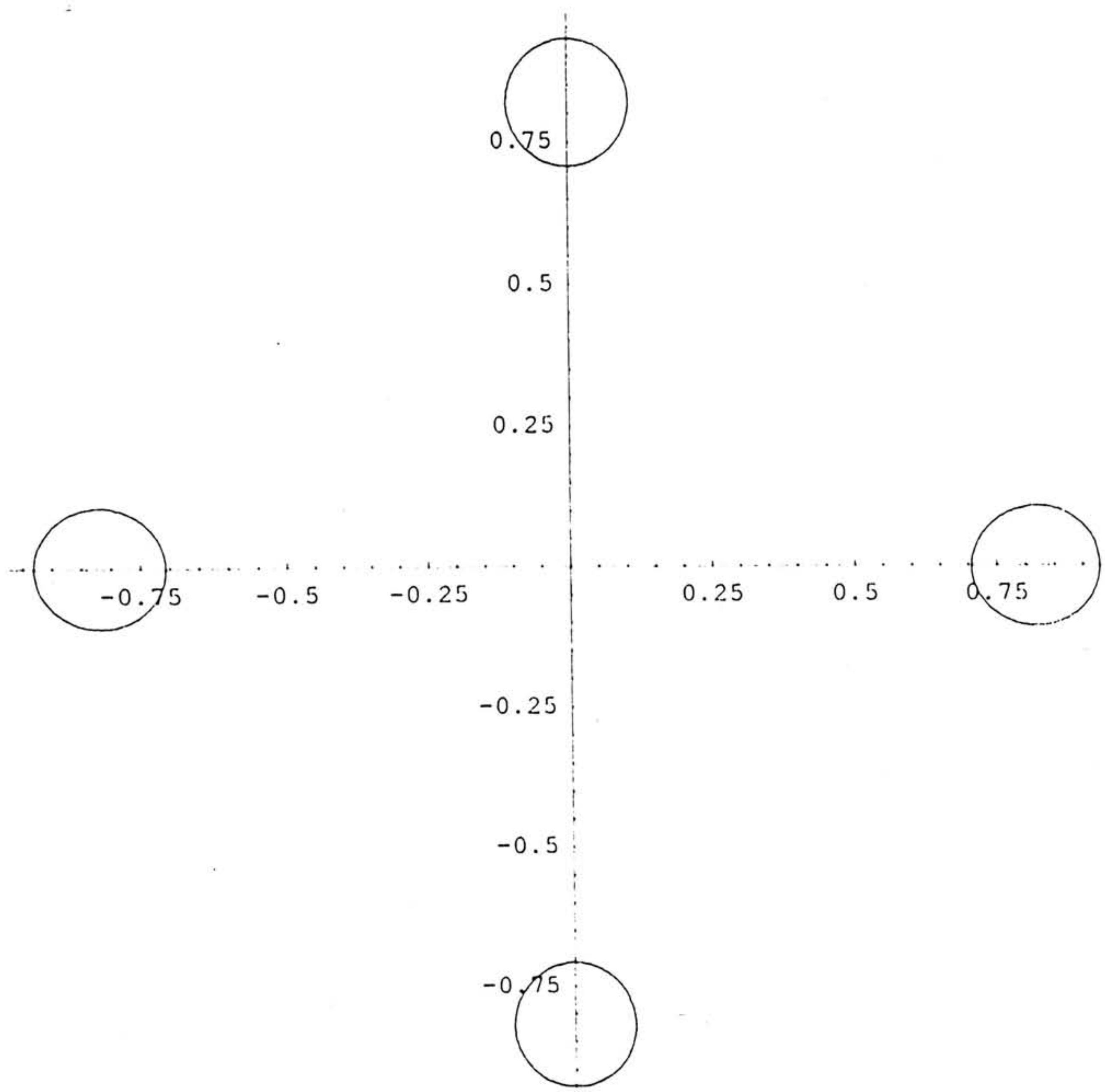


Figure 5.16 The Image of $R^{-2}(E)$



Figue 5.17 The Image of $R^{-3}(E)$

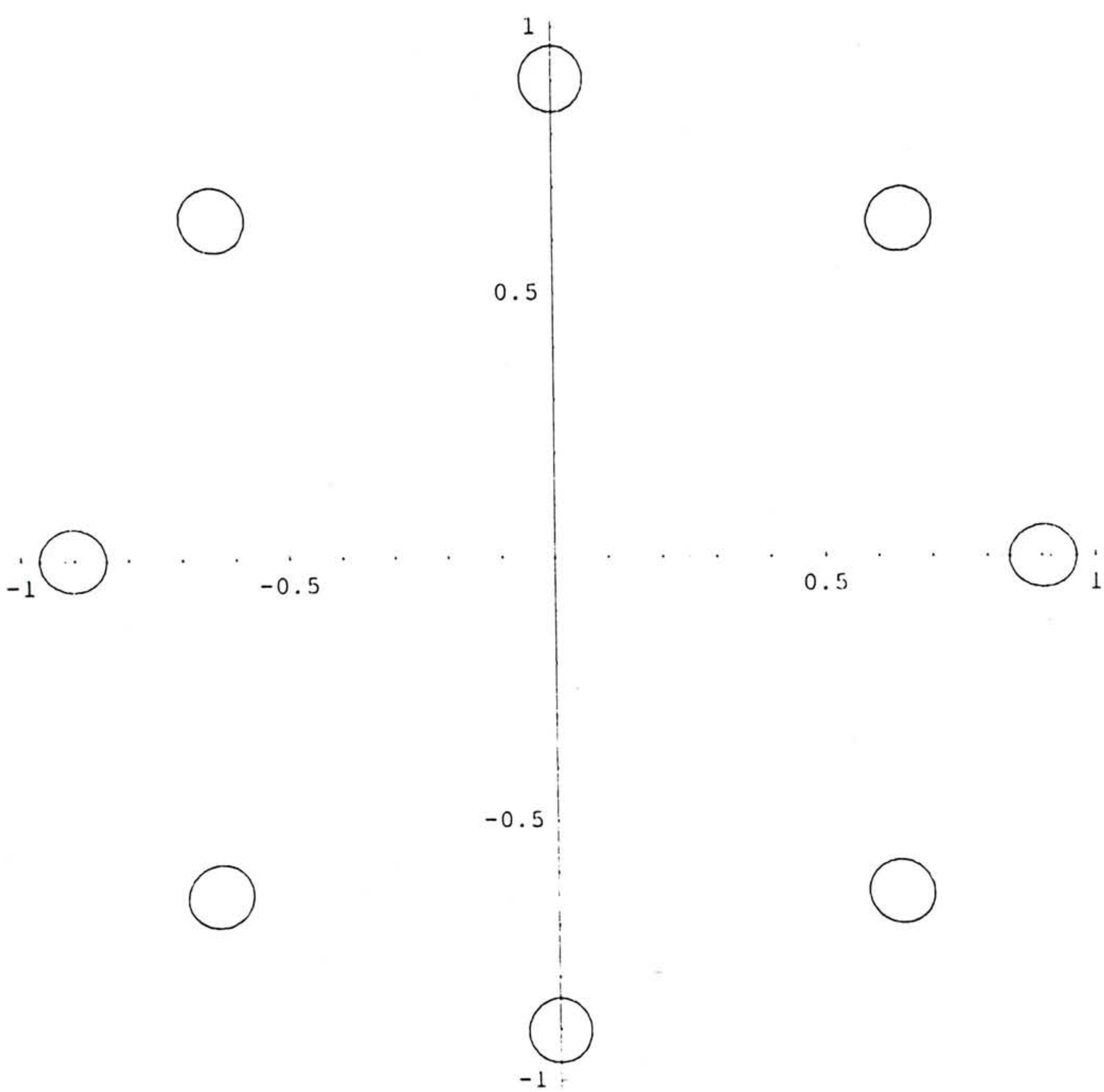


Figure 5.18 The Image of $R^{-4}(E)$

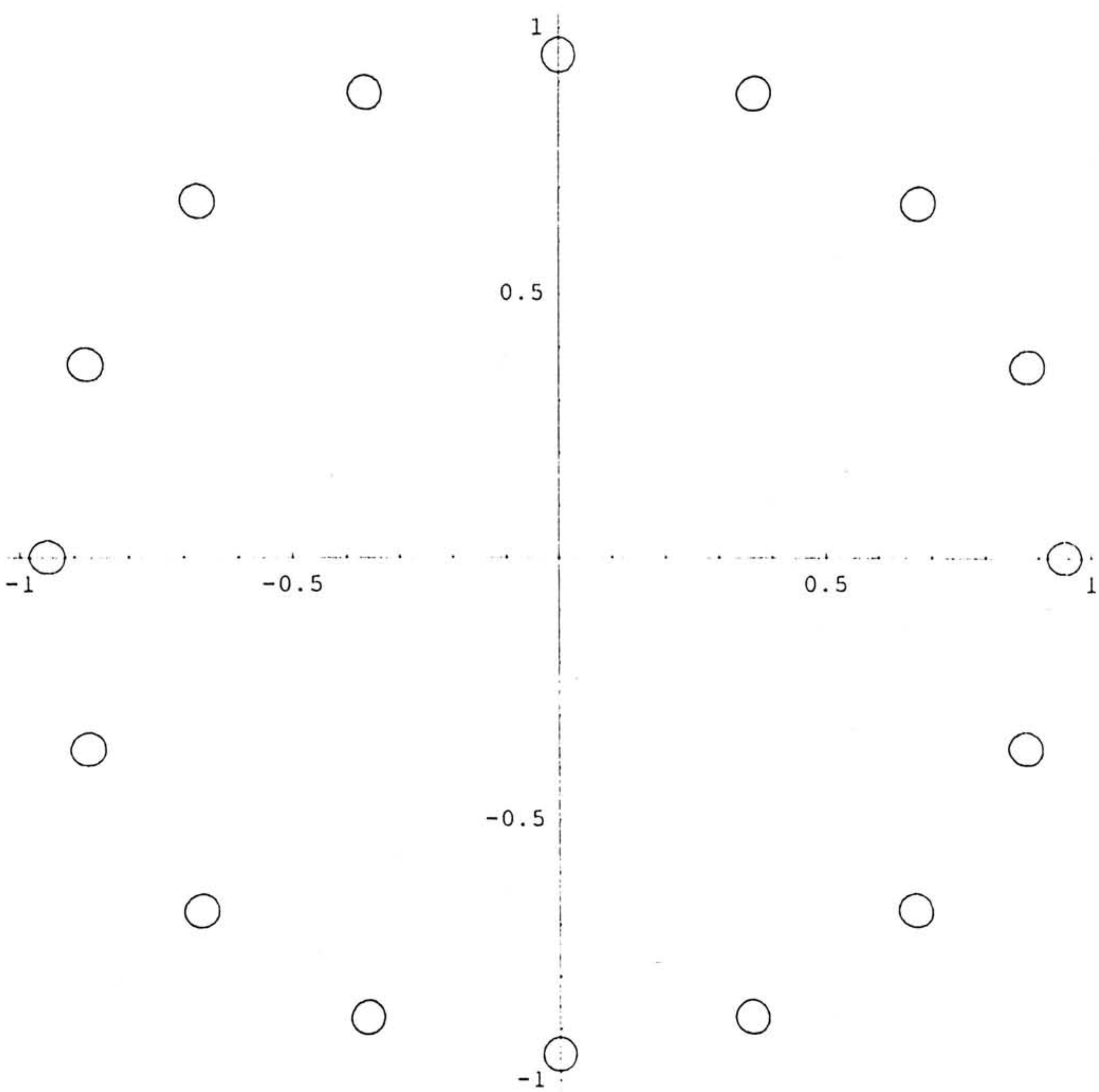


Figure 5.19 The Image of $R^{-5}(E)$

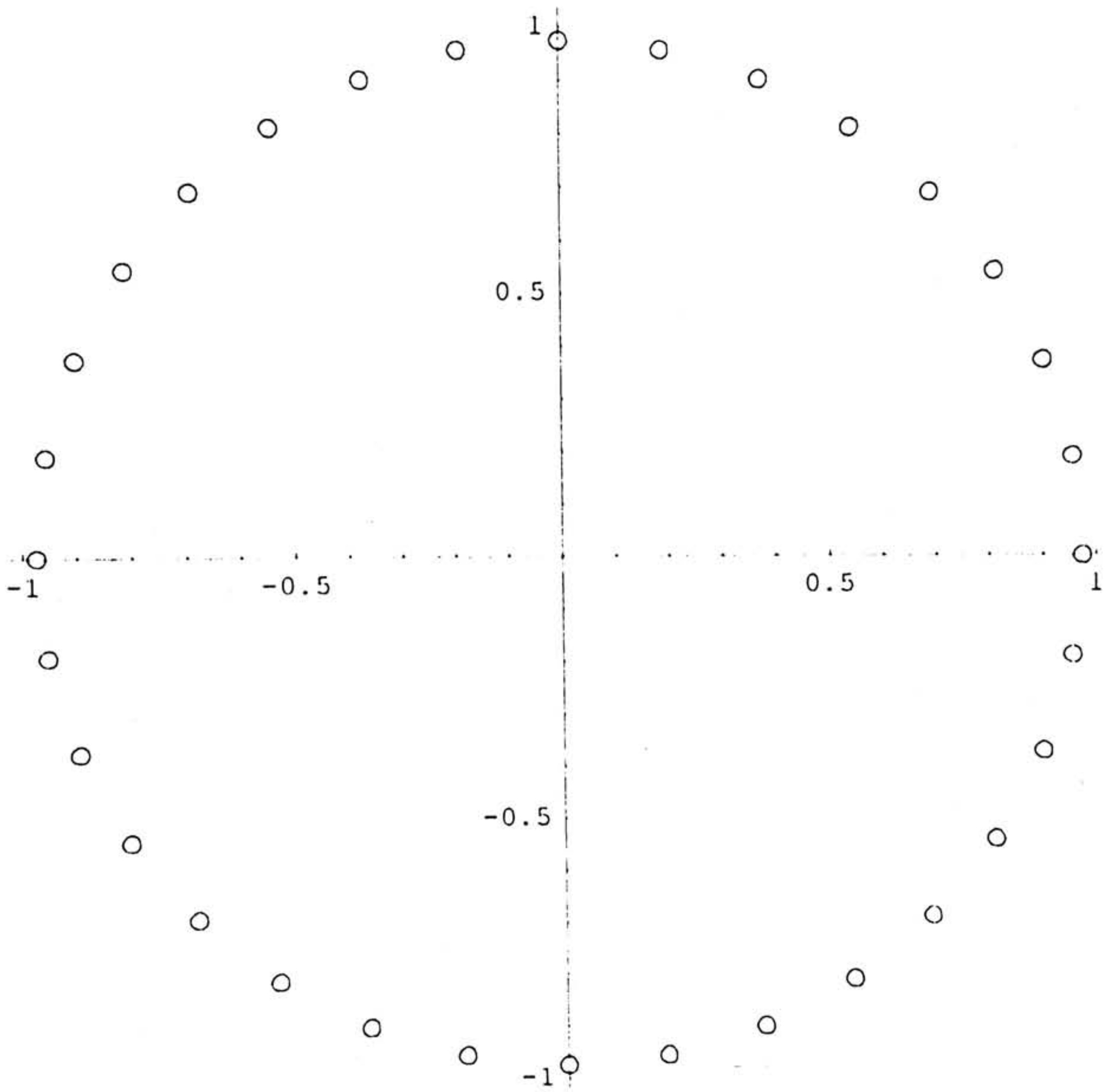


Figure 5.20 The Filled-Julia set of $z + z^4$

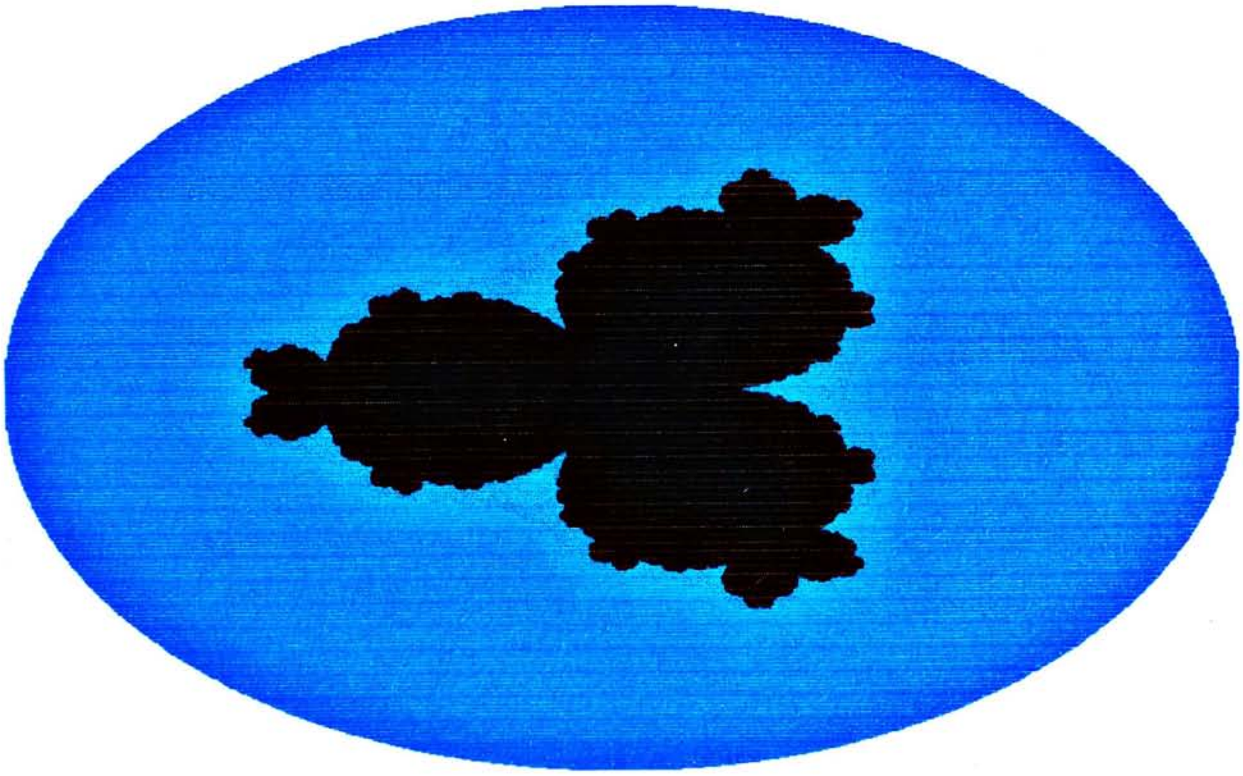


Figure 5.21 The Filled-Julia set of $\frac{3\sqrt{3}}{2}z(z+1)(z+2)$.

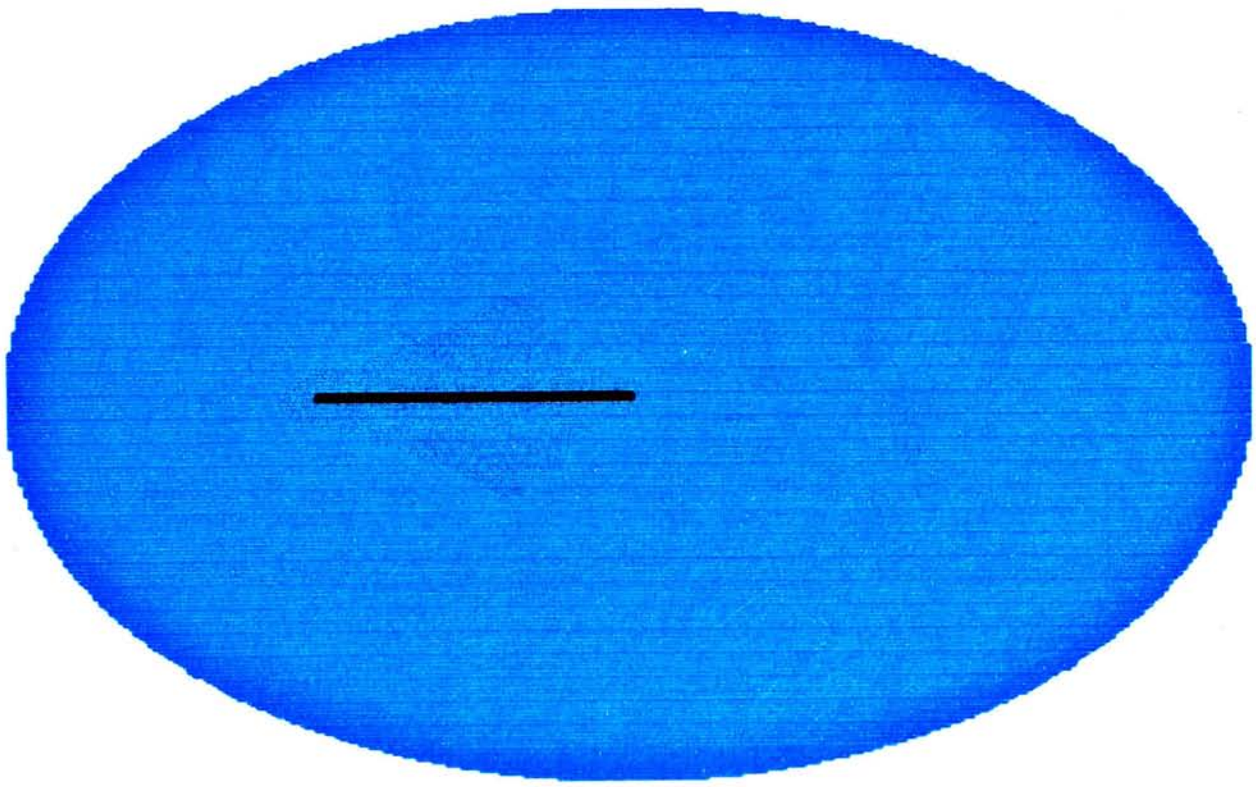


Figure 5.22 The Filled-Julia set of $z - z^4$

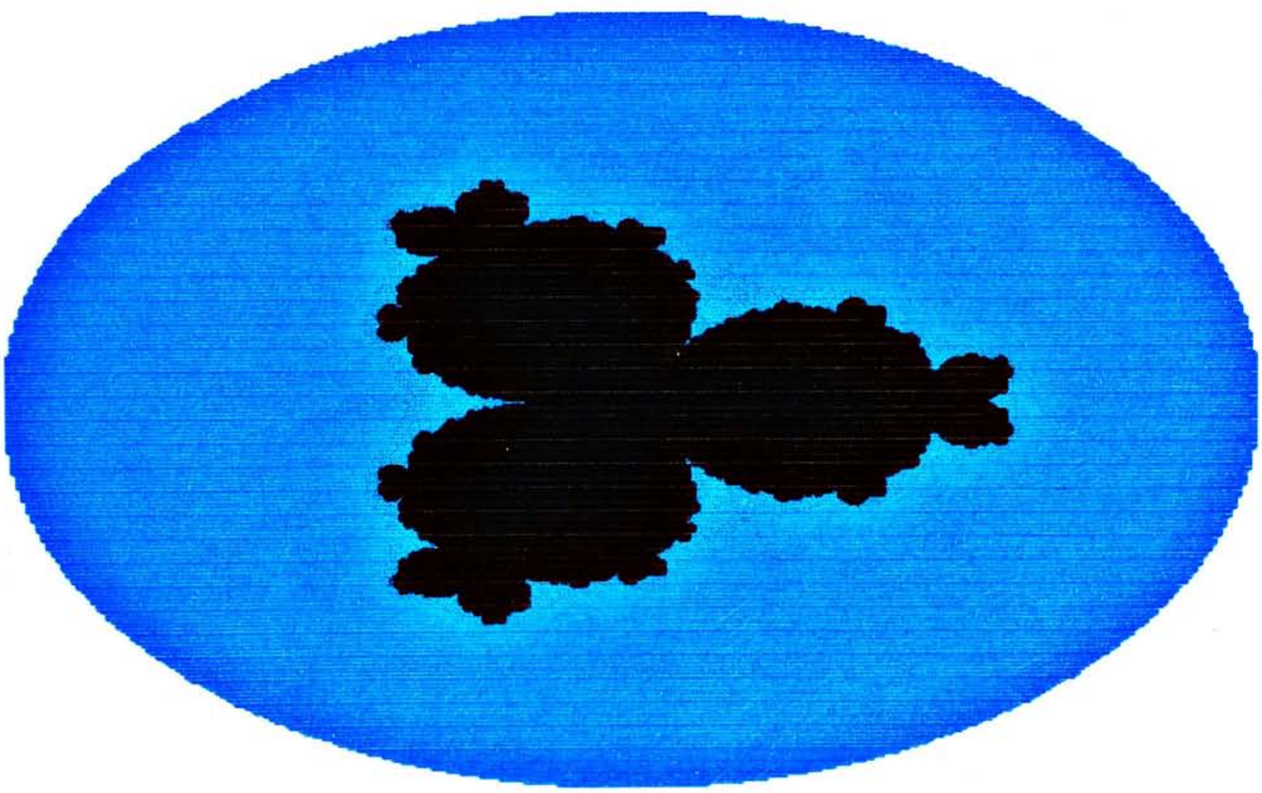
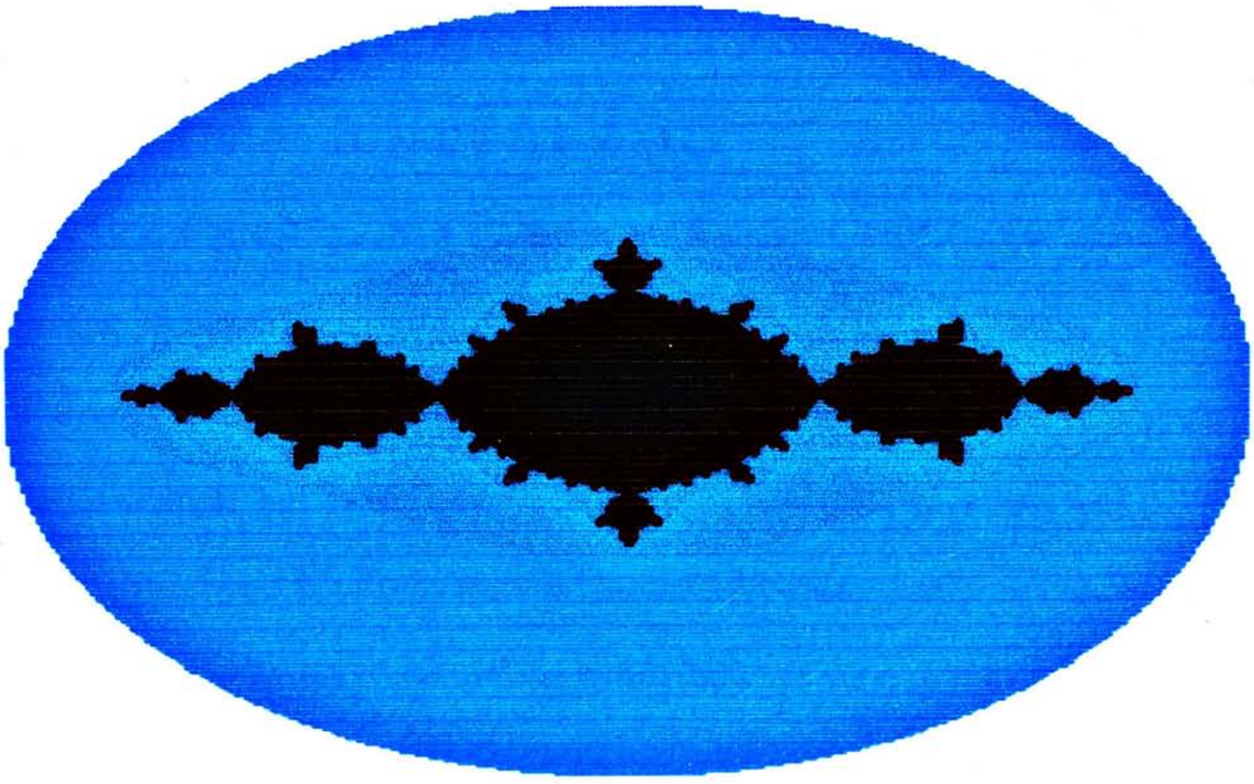


Figure 5.23 The Filled-Julia set of $z^2 - 1$.



THEOREM 5.5.8: Let U be open and suppose $U \cap J \neq \Phi$, then there is an integer N such that $R^N(U \cap J) = J$.

The above theorem is stated without proof and is illustrated by the following example.

EXAMPLE 5.5.9: There were some interesting observations on iterating a subset of the Julia set. Let R be z^2, we know that the Julia set of R is the unit circle, we choose

$$W = \{z : |z| = 1, \frac{\pi}{18} \leq \arg z \leq \frac{\pi}{9}\},$$

and let's consider $R^n(W)$ for n to be 1, 2, 3, 4, 5 and 6.

(1) $R^6(W) = J$, where $W \subseteq J$, so if we choose another subset W' of J, we will have another positive whole number N, such that $R^n(W') = J$ for $n \geq N$.

(2) W is an arc of the unit circle and the iteration of W is also an arc, though it's a very simple example, but we can observe some sort of self-similarity of Julia set.

REMARK: Theorem 5.5.8 shows that J is "almost everywhere" conformally self-similar except for those $z_0 \in J$ having a critical point in its orbit since the map does not conformal at there.

Chapter 6 Fatou Set

6.1 Components of Fatou Set

As we know from the previous chapters that Julia Set is a closed set which has empty interior or the complex sphere, therefore we can deduce that, the Fatou Set, being the complement of the Julia set , if it's non-empty, should consist of open and connected sets. From the examples given in 5.1, the Fatou set may be simply connected, which means that it consists of one component as shown in example 4.1.3, or, it may consists of two components (doubly connected) and it can be shown from our famous example of the Fatou set of $R(z) = z^2$, with the Julia set being the unit circle dividing the sphere into two components which comprises the Fatou set. Finally, the Fatou set may constitute infinitely many components, which we may call infinitely connected and can be shown from example 5.5.4. From the above observation, it may lead us to think that the Fatou set of a rational map must be simply, doubly or infinitely connected, and we shall discuss it in the coming section.

To discuss the structure of the Fatou set, F, we begin by studying the forward invariant component U of a Fatou set of a rational function, R, by recalling that $R(U) = U$, and it will be the main constituent of the following discussion in this chapter. For backward invariant component, we have $R^{-1}(U) = U$. If a component which is both forward and backward invariant, it is called a completely invariant component.

6.2 Simply connected Fatou Component

The structure of the Fatou set can be viewed from the following theorems:

THEOREM 6.2.1: Suppose that $\deg(R) \geq 2$ and that F_0 is a completely invariant component of F, then

(a) $\partial F_0 = J$

(b) F_0 is either simply connected or infinitely connected.

Proof : F_0 is completely invariant, therefore, the closure $\overline{F_0}$ of F_0 is also completely invariant, thus, by the minimality of J, $J \subseteq \overline{F_0}$. As J is disjoint from F_0, we concluded that $J = \partial F_0$.

Assume F_0 is a completely invariant component with finite connectivity c and denote the components of the complement of F_0 by E_1, ..., E_c, then there exists an integer m such that each E_j is completely invariant under R^m, since J is infinite, one of the E_j, say E_1, is infinite. The minimality of $J(R^m)$ implies that it lies in E_1 and so

$$J(R) = J(R^m) \subseteq E_1$$

However, by (a), each E_j meets $J(R)$; thus $c = 1$ and (b) follows.

In the light of the above theorem, we can see that: if $\deg(R) \geq 2$ and also F is connected, then by theorem 5.2.1(b), either F is simply connected and in this case J is connected or F is infinitely connected and J has an infinite number of components.

THEOREM 6.2.2: Suppose that $\deg(R) \geq 2$ and that F_0 is a completely invariant component of F, then

(a) all other components of F are simply connected.

(b) F_0 is simply connected if and only if J is connected.

PROOF: From theorem 6.2.1(a): $J \cup F_0$ is the closure of F_0 and so it's connected by proposition 3.4.1. By proposition 3.4.5, the components of its complement are simply connected and as these components are just the components of F other than F_0, therefore (a) follows. As $\partial F_0 = J$, by theorem 6.2.1(a) and by proposition 3.4.4, therefore F_0 is simply connected if and only if $\partial F_0 = J$ is connected.

To illustrate the theorem above, we may come back to the examples in chapter 5 which show the Julia set of a quadratic polynomial. When we consider a polynomial, P of degree greater than one, it's obvious that $\{\infty\}$ is completely invariant, therefore, the component of Fatou set of P which contains ∞, that is, the unbounded component, is also completely invariant, then by theorem 6.2.1(b), the unbounded component of the Fatou set $F(P)$ is either simply connected or infinitely connected, and it appears to be the case of our examples. Also, by theorem 6.2.2(a), all the bounded components of $F(P)$ is simply connected.

6.3 Number of components in Fatou set

DEFINITION 6.3.1: A component Ω of the Fatou set $F(R)$ is periodic if for some positive integer n, $R^n(\Omega) = \Omega$.. A *periodic domain* can be considered as a forward invariant domain of R^n for some integers, n.

In order to find out the possible number of components of a Fatou set $F(R)$, we shall discuss the number of completely invariant components first. As we know, the Fatou set is composed of components which may not be triangulated and therefore it is unlikely that the Euler Characteristic and Riemann-Hurwritz Relationship can be applied. Therefore, to begin with the discussion, we shall have a closer look at the domain we encounter and make up for the shortcoming that arises.

We note that as each invariant components are open, connected set in $\mathbf{C}_\infty$, therefore, it contains numerous closed, compact set, D_n, which can be triangulated, simply speaking D_n may be a union of a certain number of closed disks in $\mathbf{C}$. As each D_n can be triangulated, it can be used to approximate the completely invariant component D, and the Euler Characteristic of D can be defined by

$$\chi(D) = \lim_{n\to\infty} \chi(D_n).$$

It is worth to note that this definition is independent of the D_n chosen. In addition, all topological properties of D can be defined by the limit of the corresponding properties of D_n and thus, the topological theorem can be applied, for example, the Riemann-Hurwritz

Theorem. Now we can count the number of completely invariant component of the Fatou set.

THEOREM 6.3.2: The Fatou set F of R contains at most two completely invariant components and if there are two, then each is *simply connected.*

Proof: Suppose that R has degree d, where $d \geq 2$, let F_1, ..., F_k be completely invariant components of F, where $k \geq 2$. By theorem 5.2.2, we see that every components, F_j is simply connected, apply Riemann-Hurwritz relationship to each of the F_j, since R: $F_j \to F_j$, therefore

$$\delta_{R_k}(F_j) = (d-1)\chi(F_j) = d-1 \qquad \text{since } \chi(F_j) = 1$$

i.e.
$$\sum_{j=1}^{k} \delta_R(F_j) = k(d-1)$$

thus
$$\delta_R(\mathbf{C}_\infty) = 2d - 2 \geq k(d-1)$$

therefore, $k \leq 2$.

To obtain more information on the number of components in the Fatou set, Theorem 6.3.2 is not enough, since, we have no idea that for a Fatou set having 2 completely invariant components, it may have other additional components, but it can be shown that if F has two completely invariant components, then these are the only components of F. From the examples of previous chapters, we noted that F have 0, 1, 2 or infinitely many components and the following theorem tells us that these are the only possibilities:

THEOREM 6.3.3: The Fatou set of a rational map R of degree greater than 2 has either 0, 1, 2 or infinitely many components.

PROOF: Suppose that F has only finitely many components, F_1, ..., F_k, we see that each F_j is completely invariant under some iterate R^n (which has the Fatou set as R), then, according to theorem 5.3.2(applied to R^n), $k \leq 2$.

6.4 Classification of forward invariant components of the Fatou set

Let R be a rational map of degree at least two with Fatou set, F, we have the following classification for a forward invariant component F_0 of F:

DEFINITION 6.4.1: If F_0 contains an attracting fixed point ζ of R then it is a super-attracting component.

Note that both attracting and super attracting components contain an attracting fixed point in it, therefore the orbit of points in the components will tend to ζ as a limit, while the only difference is according to whether ζ is a critical point or not.

DEFINITION 6.4.2: If there is a rationally indifferent fixed point ζ of R on the boundary of F_0, and if $R^n \to \zeta$ on F_0 then it is a parabolic component.

We observe the orbit of points inside a parabolic component, it is noted that, it converges to ζ along a path which is asymptotic to the axis of the component (which according to the flower theorem (see [3]) is the straight line having a constant argument $2\pi / m$ where m is a rational number) and it agrees with our previous observation that for indifferent fixed points, orbits of points which are sufficiently close to it may move away or towards it and the above observation is to be illustrated in the following section.

By considering the limit of some subsequence of R^n in F_0, we can see that : for attracting components, super-attracting components and parabolic components, the only possible limit is the fixed point ζ, since any subsequence of R^n in F_0 converges to ζ The only difference for the three components is that: ζ may lie on the boundary of the component and it may be critical. However, if F_0 is forward invariant and if there is a constant limit function, φ with value ζ, then ζ is a fixed point of R, since

$$R(\zeta) = R(\lim_{n\to\infty} R^n(z)) = \lim_{n\to\infty} R^n(R(z)) = \varphi(R(z)) = \zeta$$

DEFINITION 6.4.3: If R: $F_0 \rightarrow F_0$ is analytically conjugate to a Euclidean rotation of the unit disc onto itself then it is an Siegel disc.

DEFINITION 6.4.4: If R: $F_0 \rightarrow F_0$ is analytically conjugate to a Euclidean rotation of some annulus onto itself then it is called a Hermen ring.

The orbit of any point, ω in the Siegel disc and Herman ring is conjugate to a circle around the conjugate of the fixed point while the two components can be distinguished either by the connectivity of F_0, or by the existence of a fixed point in F_0. It is worthwhile to notice that in both components, they have non-constant limit function.

To summarize, a forward invariant component of R can be one of the above and they can be distinguished by whether they have constant or non-constant limit function in the component and the position of the fixed point and critical point. For attracting and super-attracting component, the fixed point and critical point lie inside the component. For parabolic component, the fixed point lies on the boundary of the component and the critical point lie inside the component, while for Siegel disc and Herman ring the fixed point lie inside the component and the critical point lie on the boundary.

In the final section of this chapter, we discuss some examples of the component defined.

6.5 Examples illustrating the five possible forward invariant components

In the final section of this chapter, we discuss some examples of the component defined. In the following examples, the orbits of points in a forward invariant component will be shown together with the fixed point and the critical point and if possible, discussion on the limit may also be included. It is aimed at giving a clearer picture of what will happen when iteration being carried out inside these components.

EXAMPLE 6.5.1: We first come to our well-known examples of the forward invariant components of $R(z) = z^2$, it is easily shown that 0 is a super-attracting fixed point and the forward invariant component is the unit disk with centre at the origin, it can be shown from Figure 6.1 that, both the fixed point and the critical point 0 lie in the unit disk. Also the orbit of any point in the component converges rapidly to the origin, the fixed point, therefore, the unit disk is a super-attracting component of $R(z) = z^2$. It is also known that $R^n(z) = z^{2^n}$, for any compact set $K \subseteq F_0$ i.e. the unit disk, take $z \in K \subseteq B(0; r)$ where $r < 1$, then we have $|R^n(z)| = |z^{2^n}| < r^{2^n}$ which tend to zero as n tends to infinite, therefore R^n converges uniformly to 0 on K, that is, there is only one limit function in F_0 which is the constant function with value, zero.

EXAMPLE 6.5.2: Consider the polynomial $R(z) = z - z^2$, Figure 6.2 shows the filled-in Julia set of the polynomial and also the fixed point 0 which is in red while the critical point 0.5 is in green. It should be emphasized that the origin lies on the boundary of the Fatou component while the critical point 0.5 lies inside the Fatou component, since it's easy to check that 0 is a neutral fixed point and hence the Fatou component can be classified as a parabolic component.

Figure 6.1 The Filled-Julia set of $R(z) = z^2$ and its critical point

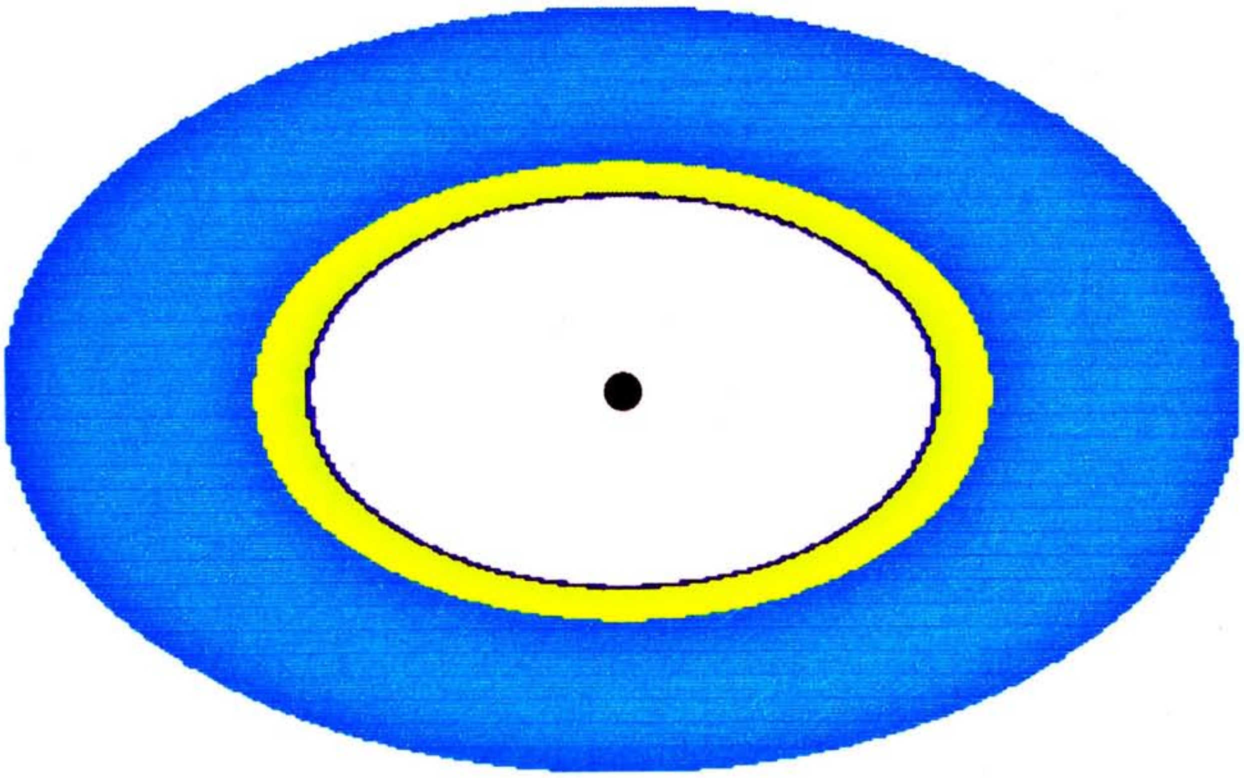
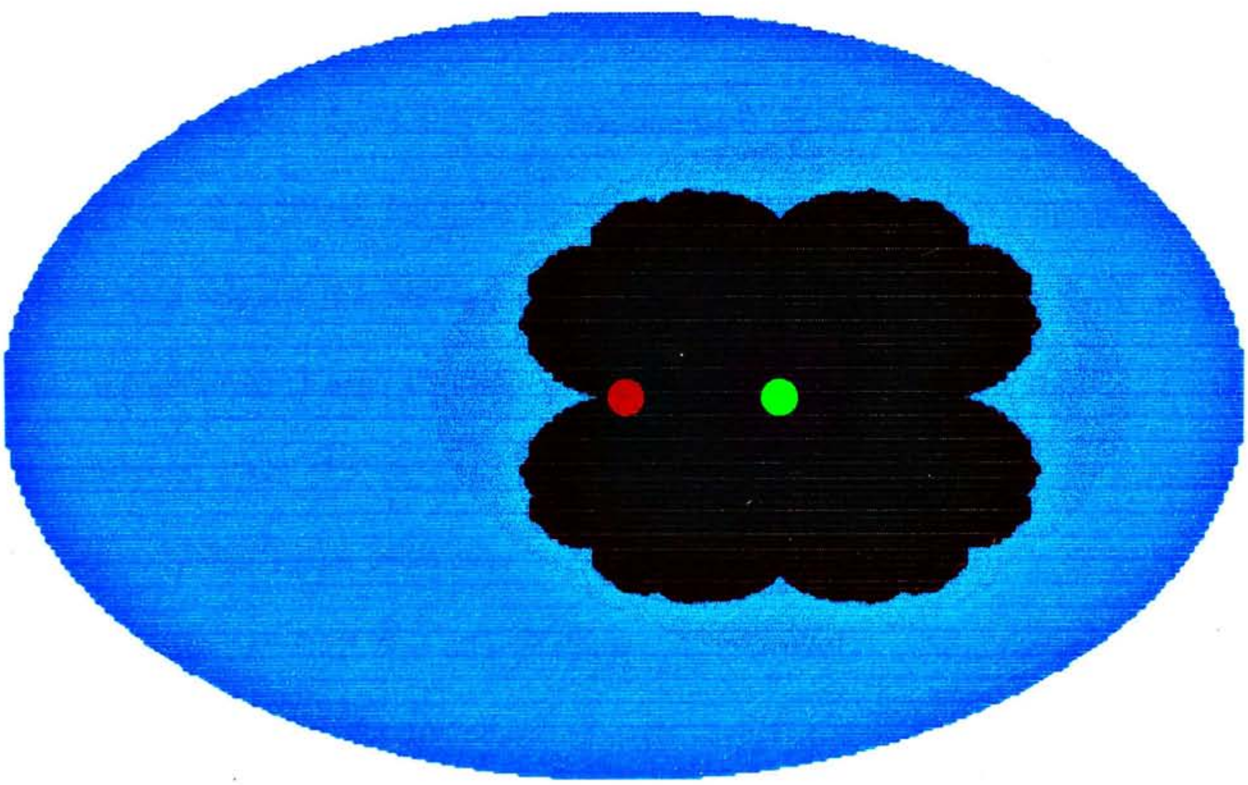


Figure 6.2 The Filled-Julia set of $R(z) = z - z^2$, its critical point and fixed point.



Chapter 7 Critical Point

7.1 Introduction

In this chapter, we are going to discuss the influences of critical points of a rational map on the general feature of the global dynamics of the map. As we know that a rational map, R is not injective in any neighbourhood of its critical points and the influences of these points on the dynamics of R largely results from the Riemann-Hurwritz relation, recalled as that: R has at most $2d - 2$ critical points where d is the degree of R. In the case of polynomial P of degree d, where $d \geq 2$, our attention should be directed to the $d - 1$ finite critical points of P, that is, for those which lie on $\mathbf{C}$. In considering the location of the critical points of a rational map of degree at least two, it is interesting to note from the examples of the previous chapter that the immediate basin of each (super) attracting cycle, rationally indifferent cycle of R contains a critical point of R, it means that each attracting cycle and rationally indifferent cycle attracts an infinite forward orbit of some critical point of R and in the case of Siegel disc and Herman ring, the closure of the forward orbit of some critical point contains the boundary of each Siegel disc and also each boundary component of each Herman ring. The above information together with the Riemann-Hurwritz relation gives us some idea of the number of those cycles mentioned above.

Before we proceed to the application of the results of the above section, we shall introduce an important theorem by Sullivan in 1983 which concerned with the sequence of components by acting a rational map R on any component Ω of the Fatou set of R, i.e. Ω, $R(\Omega)$, $R^2(\Omega)$, ...

DEFINITION 7.1.1: A component Ω of the Fatou set $F(R)$ is:

(a) *periodic* if for some positive integer n, $R^n(\Omega) = \Omega$

(b) *eventually periodic* if for some positive integer m, $R^m(\Omega)$ is periodic;

and (c) *wandering* if the set $R^n(\Omega)$, $n \geq 0$, are pairwise disjoint.

Also, a *wandering component* of $F(R)$ is called a *wandering domain.*

THEOREM 7.1.2 (Sullivan No Wandering Domain Theorem): Every component of the Fatou set of a rational map is eventually periodic.

Sullivan's result is important since it implies that when considering the long term dynamics of a rational map R on its Fatou set F means to consider the action of R on the periodic components of F, as we know that a periodic component of F is forward invariant under some R^m, where m is an integer, but then, by the discussion in previous chapter, these components can be classified into 5 categories which combined with the results on the critical points in the above section: periodic components imply the existence of critical points and from the Riemann-Hurwritz relation, we know that there are at most $2d - 2$ critical points, thus, more quantitative information and some interesting results can be obtained.

Some proofs of the results that follows may involve the application of Sullivan's No Wandering Domain Theorem, since the proof of the captioned theorem required some advanced technique, which will not be included in the discussion, and thus, the result is assumed without proof(readers who are interested may read [3]).

7.2 Some Interesting Results

From the discussion of the above section, it is rather obvious that a rational map R of degree d has at most $2d - 2$ (super)attracting cycles together with rational indifferent cycles, since we knew that distinct cycles have disjoint basins and therefore it is a direct consequence of the Riemann Hurwritz relation and the location of the critical points. The above reasoning seems direct and natural but the result obtained is rather impressive since R has infinitely many cycles, and thus we have the following theorem:

THEOREM 7.2.1: Let R be a rational map of degree d, where $d \geq 2$. Then the combined number of (super)attracting and rationally indifferent cycles is at most $2d - 2$.

To get more information, let's consider a quadratic polynomial as our specific case: there are only two critical points with one of which lies in **C**, while the other being ∞, therefore, it has at most two (super)attracting and rationally indifferent cycles with one of which contains ∞, that is, the completely invariant component, F_∞. If the forward orbit of the finite critical point is attracted towards ∞, then there cannot be another such component, so $F(R) = F_\infty$ and $F(R)$ is connected. It is natural to generalize the result for polynomial of higher degree: Let's consider the case of a cubic polynomial, there are altogether 3 critical points for the cubic polynomial with one of them being ∞, there is no doubt that if both critical points which belong to **C** attract to ∞, then $F(R)$ is connected, the argument is the same as that in the quadratic case. For the case of cubic polynomials, some complications arise: what will be the general feature of $F(R)$ if one or both of the finite critical point(s) do not attract to ∞? Before we answer the question, let's take a look at the Julia set of some rational map first.

EXAMPLE 7.2.2: Consider the Julia set of the polynomial $P(z) = z^2 - \frac{z^3}{9}$, the critical points of $P(z)$ are 0, 6 and ∞ and $R^n(0) = 0$ and $P^n(6) \to \infty$, that is one of the critical point being attracted to ∞ while the other does not. We know that F_∞ is infinitely connected, also F has infinitely many other components and each being simply connected and J has infinitely many non-degenerate components.

EXAMPLE 7.2.3: Cosider the Julia set of the polynomial $P(z) = z^3 - 12z^2 + 36z$, the critical points of $P(z)$ being 2, 6 and ∞, furthermore, $P^n(6) = 0$ and $P^n(2) \to \infty$. We can see that F is connected and is of infinite connectivity; also J is a Cantor set.

From the above examples, we can observe that for cubic polynomials: if one of its finite critical point attract to ∞, then the Fatou set of the polynomial is infinitely connected, with its Julia set either being a Cantor set or it may contain infinitely many non-degenerate components, while if both of them are attracted to points in **C**, also the orbits of the critical points does not belong to Julia set, we see that the Fatou set is infinitely connected with components also infinitely connected, but in fact, if one or both of the orbit(s) of the critical point lie in the Julia set, we have no conclusion about it.

THEOREM 7.2.4: If the Fatou set $F(R)$ of R has 2 completely invariant components, then they are the only components of $F(R)$.

PROOF: By the assumption, there exists 2 completely invariant components, say F_1 and F_2. Now neither F_1 nor F_2 can be a Seigel disc or a Herman ring, otherwise R could be a injective map of F_j onto itself, then by completely invariance, $\deg(R) = 1$. It follows that each F_j is an attractive , a super attractive, or a parabolic component of $F(R)$ and in each of these cases, the forward orbit of any point in F_j converges to a fixed point ζ_j in the closure of F_j. Now every component of F is simply connected and in particular, F_1 and F_2 are. By considering the map R of F_j onto itself, together with simply connectivity and completely invariance of each F_j, we obtain

$$\delta(F_1) = d - 1 = \delta(F_2)$$

where $\deg(R) = d$. This, together with the fact that R has at most $2d - 2$ critical points, it implies that the forward orbit of every critical point lies in $F_1 \cup F_2$ and accumulates only at ζ_1 or ζ_2.

Suppose now that F had other components: then it would necessarily contain some cycles of components disjoint from the invariant components F_1 and F_2. Now by the discussion of 7.1, such a cycle would either have to meet some forward orbit of critical points and it cannot because F_1 and F_2 contain all of the critical points of R, or it would have to contain Siegel discs or Herman rings. In this latter case, there would be infinitely many accumulation points of forward orbits of critical points, and as these forward orbits accumulate only at ζ_1 and ζ_2, this case cannot occur either. We have now eliminate all possibilities, so $F(R)$ has no other components.

THEOREM 7.2.5: If every critical point of R is pre-periodic, then $J(R) = \mathbf{C}_\infty$.

PROOF: We suppose that every critical point of R is pre-periodic and that F is non-empty; and we seek a contradiction. As F is non-empty, then by Sullivan's No Wandering Domain Theorem, there is some component of F which is periodic under the action of R. Now such a cycle of components is associated with a super-attracting cycle, an attracting cycle, a rationally indifferent cycle, a cycle of Siegel discs or a cycle of Herman rings. In the first case there is some periodic critical point, while in the remaining cases, there is a critical point with an infinite forward orbit. By assumption, none of these cases can arise so F must be empty and $J = \mathbf{C}_\infty$.

In the previous chapter, we know that the rational map $R(z) = \dfrac{(z-2)^2}{z^2}$ has empty Fatou set, we now give it a proof: the critical points for R are 2 and ∞, but

$R(2) = 0$, $R(0) = \infty$, $R(\infty) = 1$ and $R(1) = 1$, i.e. 2 is pre-periodic also $R(\infty) = 1$ and

$R(1) = 1$, therefore, ∞ is also pre-periodic and by Theorem 7.2.6, $J(R) = \mathbf{C}_\infty$.

7.3 The Fatou Set of a Polynomial

From previous discussion, we know that every bounded component of Fatou set of a non-linear polynomial is simply connected and that the unbounded component F_∞ of Fatou set of a polynomial is either simply connected or infinitely connected and the following results show that the dynamics of the finite critical points of a polynomial determines the connectivity of F_∞.

THEOREM 7.3.1: Let P be a polynomial with $\deg(P) \geq 2$, then the followings are equivalent:

(a) F_∞ is simply connected;

(b) J is connected;

(c) there are no finite critical points of P in F_∞.

PROOF: We already knew that (a) and (b) are equivalent. We are going to prove that (a) $\Leftrightarrow$ (c). Assume that F_∞ is simply connected i.e. $\chi(F_\infty) = 1$. Applying the Riemann-Hurwritz relation to the map P of F_∞ onto itself, we obtain

$$1 + \delta(F_\infty) = d,$$

and as P has deficiency $d - 1$ at ∞, there cannot be any finite critical points of P in F_∞; thus (c) follows.

Assume that there are no finite critical points in F_∞. From this and the complete invariance of F_∞, we can see that P^n has no finite critical points in F_∞. Now find a disc centred at ∞ and such that

$$P(D) \subseteq D \subseteq F_\infty,$$

and define $D_0 = D$, and $D_n = P^{-n}(D)$. Then, as before, each D_n is a domain containing ∞, and the D_n' s satisfy

$$D = D_0 \subset D_1 \subset D_2 \subset \dots$$

Applying the Riemann-Hurwritz relation to the map P^n of D_n onto D, and noting that there are no finite critical points of P^n in F_∞, we find that

$$\chi(D_n) + (d^n - 1) = \chi(D_n) + \delta(D_n) = d^n \chi(D_n)$$

Now D is simply connected, so $\chi(D_n) = 1$ and D_n is simply connected. As F_∞ is the union of the increasing sequence of simply connected domains D_n , therefore, it too, is simply connected and the proof is completed.

Viewed from theorem 7.3.1, we have the following corollaries and we'll give a few examples to illustrate these corollaries.

COROLLARY 7.3.2: If either (a) every finite critical point of P lies in J or

(b) every finite critical point of P has a finite orbit , then

F_∞ is simply connected.

COROLLARY 7.3.3: If every finite critical point of P is pre-periodic, then F is connected and simply connected.

EXAMPLE 7.3.4: As examples of the three hypotheses in the above corollaries, we noted that for $z \mapsto z^2 - 2$, $0 \in J$ and we noted from figure 5.10 that F_∞ is simply connected; for $z \mapsto z^2 - 1$: 0 → -1 → 0 → -1 →... which shows that 0 has a finite forward orbit, from figure 5.11, F_∞ is simply connected; for $z \mapsto z^2 + i$, note that $0 \to i \to -1 + i \to -i \to -1 + i \to \ldots$ which is preperiodic, from figure 5.13, F is connected and simply connected.

7.4 Quadratic Polynomial and Mandelbrot Set

Throughout this chapter, effort was put on the studying the properties of the Fatou set and the Julia set which aimed at finding a general trend and structure of the Julia set of a rational map, may be more precisely, a polynomial. The discussion in previous sections had shown the importance of the role of critical points on the structure of the Fatou set. Difficulties arise in discussing the structure of the Fatou set when degree of the polynomial greater than two, since there will be more finite critical points and more complications when the orbit of the critical points show different behaviour, especially for those cases which involving Siegel disk or Herman ring.

Fortunately, for quadratic polynomial, $f(z) = az^2 + bz + c$, when undergo suitable conjugation (see section 1.5), can be expressed in a simple form: $P(z) = z^2 + k$. From the discussions of previous chapters: P and f have the same Julia set and the structure of the Julia set may be revealed by studying the orbit of the only finite critical point, i.e. the origin. From the results of 7.1 and 7.2, the Julia set is either connected or totally disconnected which can be determined by the following theorems:

THEOREM 7.4.5: The Mandelbrot set M is a closed simply connected subset of the disk $\{ |k| \le 2\}$, which meets the real axis in the interval [-2, 1/4]. Moreover, M consists of precisely those k such that $|P_k^n(0)| \le 2$ for all $n \ge 1$.

The above theorem will be stated without proof, (for those who are interested, see [6]) and the Mandelbrot set is illustrated in Figure 7.1. The importance of the Mandelbrot set is that the shape of the Julia set of a quadratic polynomial depends on the position of k in the Mandelbrot set and will be illustrated in the following figures. From Figure 7.2 to 7.6, we will show the shape of the Julia set with real values of k varies from -1.5 to 1.2. We can see that the shape of the Julia set change drastically from one value to another.

THEOREM 7.4.1: The Julia set is connected if and only if there is no finite critical point of P in $A(\infty)$, the attractive basin of ∞, that is, if and only if the forward orbit of each finite critical point is bounded.

THEOREM 7.4.2: If $P^n(q) \to \infty$ for each critical point of q, then the Julia set is totally disconnected.

In section 5.5, we have shown some examples of Julia set of the quadratic polynomials. The shape of the Julia set can also be observed to be dependent on the parameter k. Let $P_k = z^2 + k$, we are interested in how the dynamic behaviour of P_k depends on the parameter k. Firstly, Theorem 7.4.1 and 7.4.2 can be rewritten as:

THEOREM 7.4.3: If $P_k^n(0) \to \infty$, then the Julia set is totally disconnected, otherwise, $P_k^n(0)$ is bounded and the Julia set is connected.

DEFINITION 7.4.4: The Mandelbrot set, $\mathcal{M}$, is defined as $\mathcal{M} = \{k: P_k^n(0) \text{ is bounded}\}$

From definition 7.4.4, we see that $k \in \mathcal{M}$ if and only if O does not belong to the basin of attraction of the super attracting fixed point at ∞.

The following theorem suggests a simple algorithm for computing $\mathcal{M}$:

Figure 7.1 The Mandelbrot set

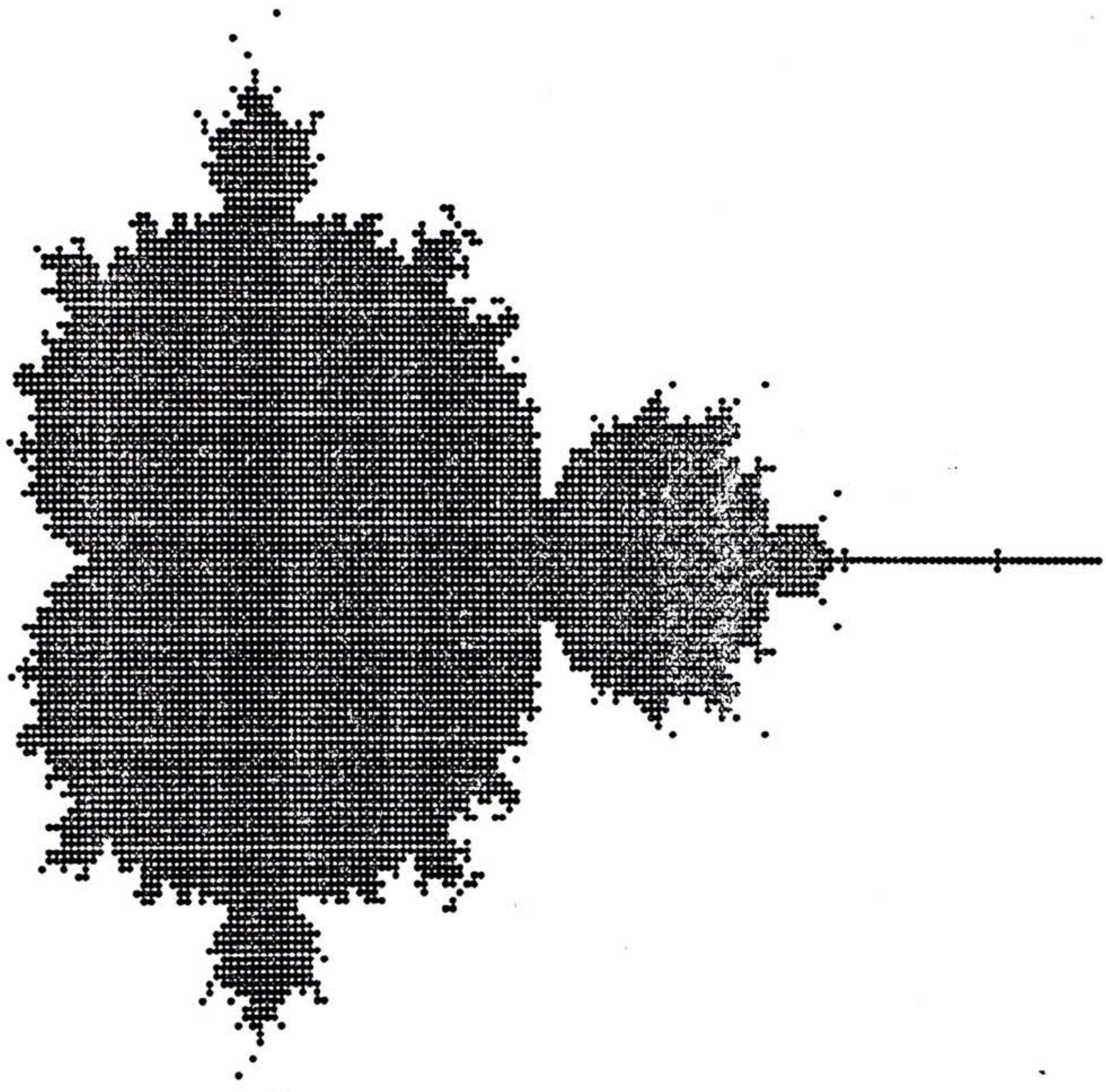


Figure 7.2 The Julia set of $R(z) = z^2 - 1.5$.

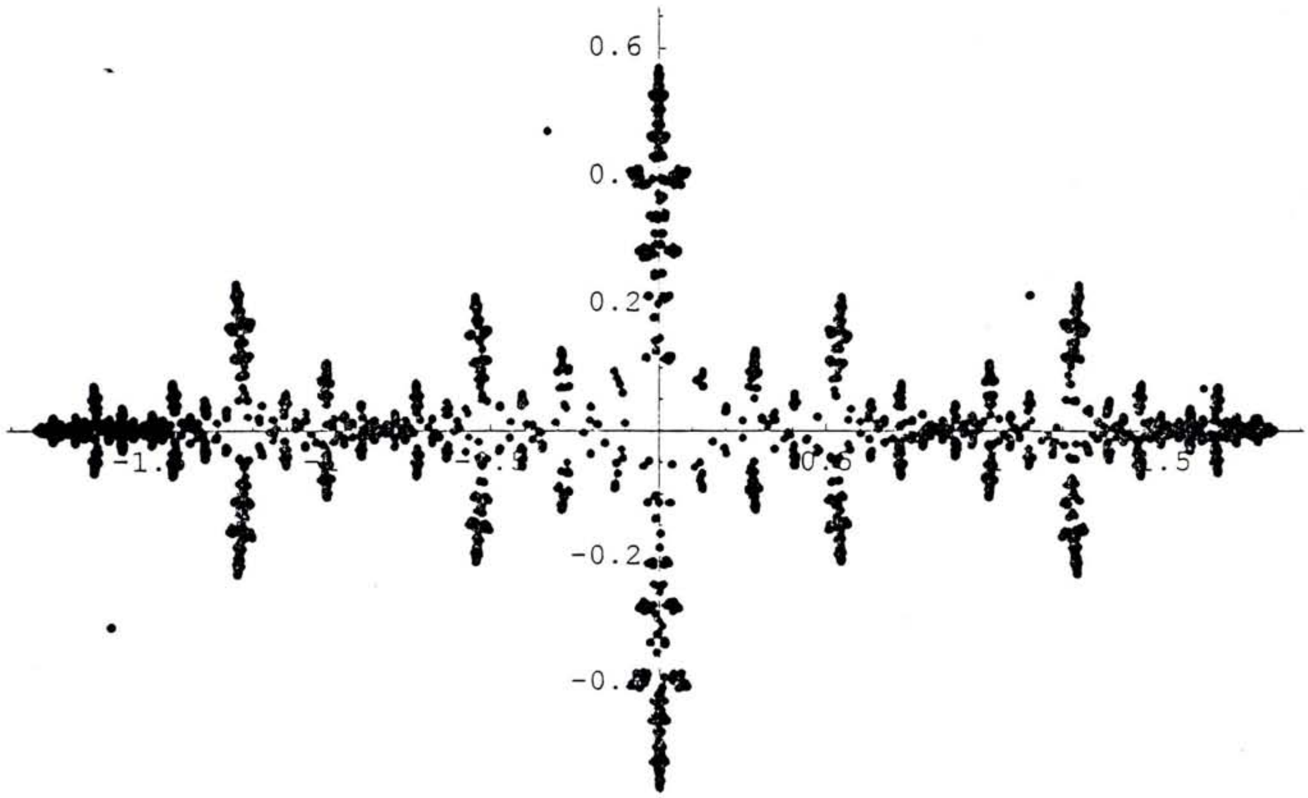


Figure 7.3 **The Julia set of $R(z) = z^2 - 1.1$.**

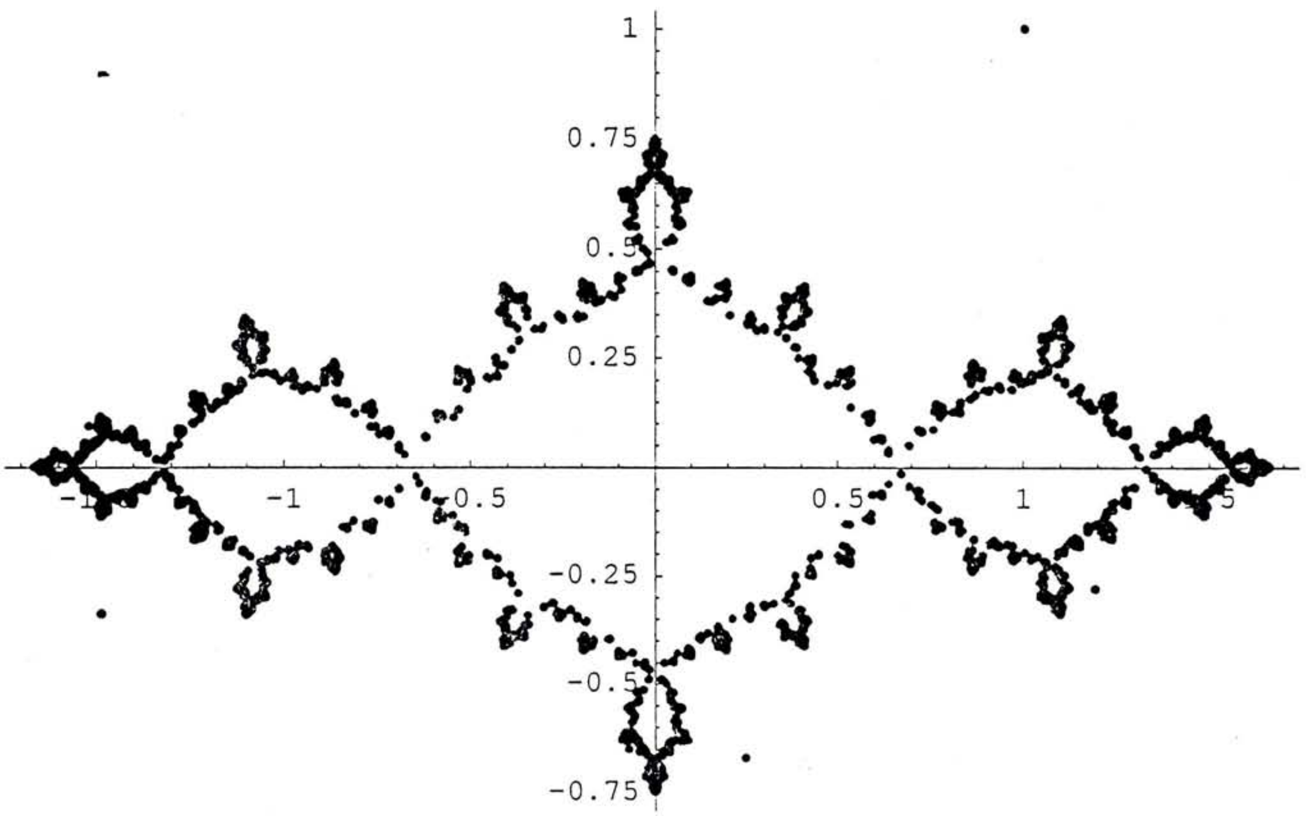


Figure 7.4 The Julia set of $R(z) = z^2 - 0.25$.

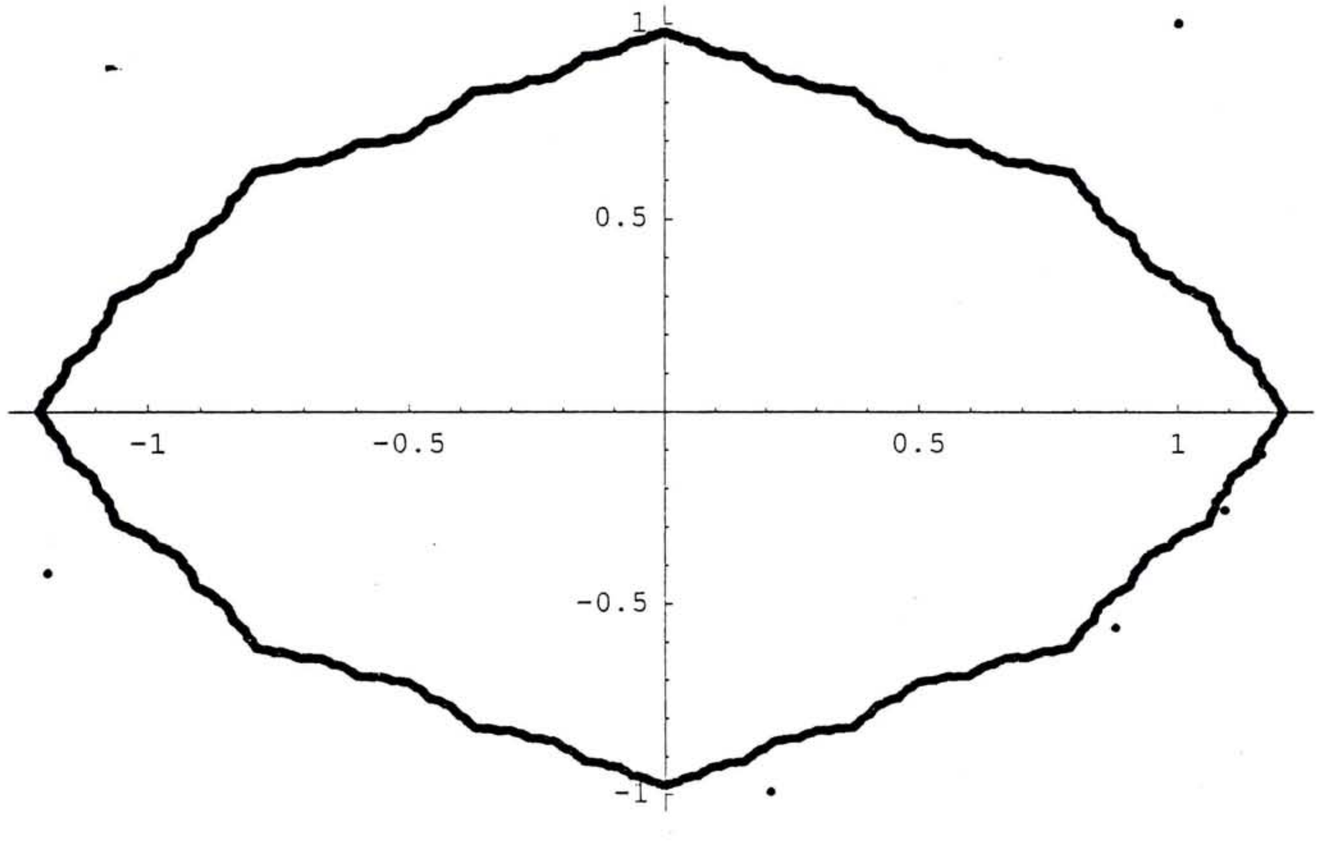


Figure 7.5 The Julia set of $R(z) = z^2 + 0.251$.

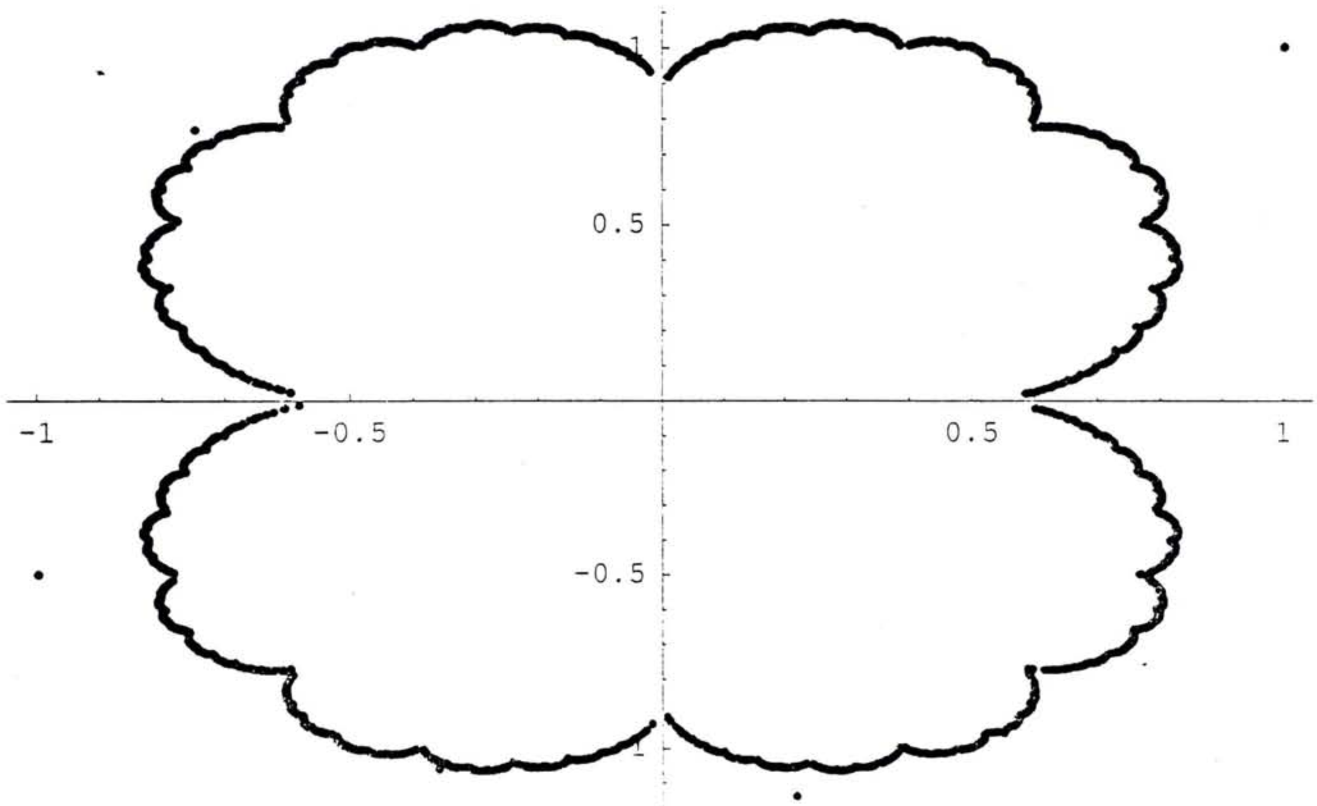
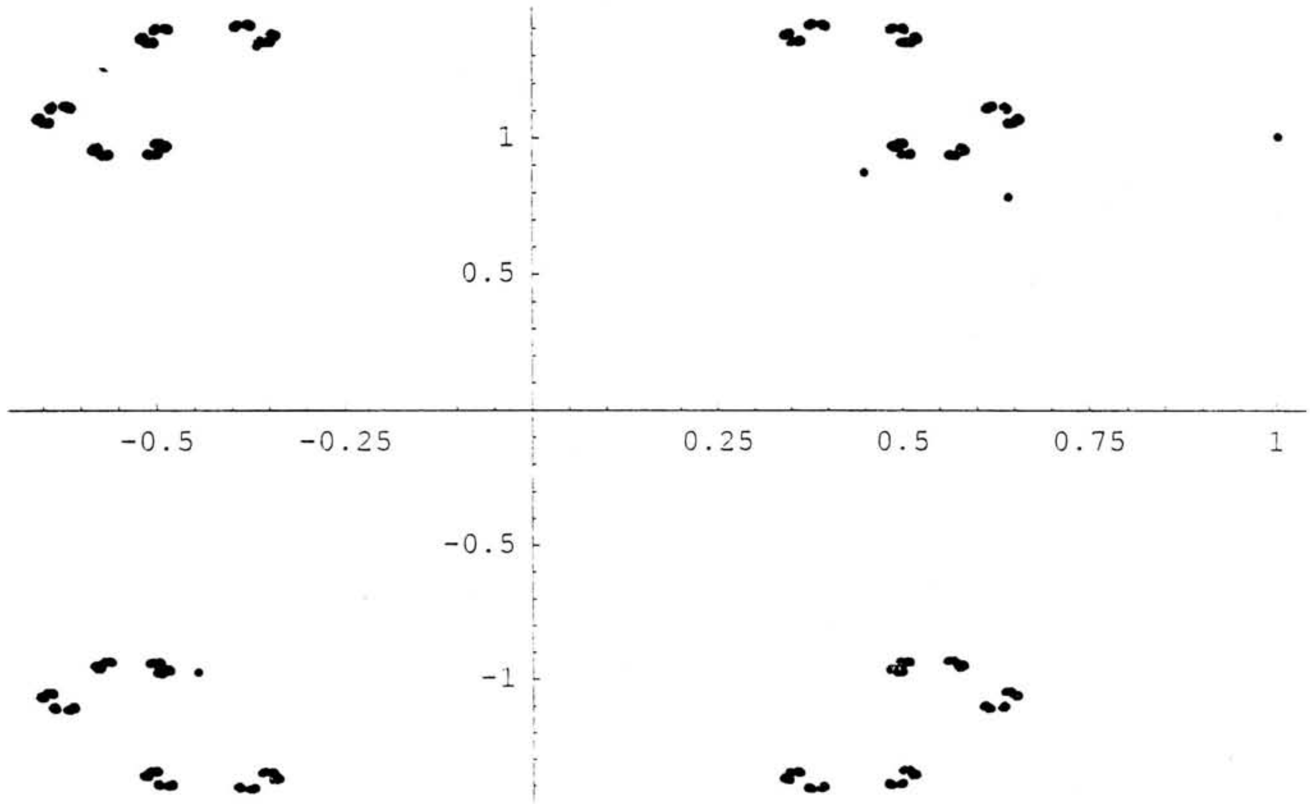


Figure 7.6 The Julia set of $R(z) = z^2 + 1.2$.



Appendix

Programming codes written in Mathematica to illustrate the computer graphics of complex dynamics will be included in this appendix. They will be divided into five categories:

1. Programme codes used to show the orbits of some chosen points:

In order to show the order of iteration of the points in an orbit, a gradual change in colour (from lighter to darker) of the points had been designed. If there were more than one orbits shown in the same graph, different colours had been adopted. The function (`f`), initial point (`z`) and the number of iterations done (`n`) can be chosen arbitrarily. The programme codes are shown as follows:

```
Orbit[f_,z_,n_]:=Block[{w,listI,k},w=z;
                       listI=Table[RGBColor[1,0,1],
                                   PointSize[.08],
                                   Point[{Re[z],Im[z]}]}
                                  ];
                          k=1;While[k<=n,{w=f[w];
                             AppendTo[listI,
                                    {CMYKColor[0,1,0,k/n],
                                    PointSize[.08-0.07k/n],
                                    Point[{Re[w],Im[w]}]}];
                                    k++}
                                   ];
                             Show[Graphics[listI,
                                    Axes->True,
                                   AspectRatio->1
                                          ]
                                   ]
                       ]

RedOrbit[f_,z_,n_]:=Block[{w,listI,k},w=z;
                       listI=Table[{RGBColor[1,0,1],
                                   PointSize[.08],
                                   Point[{Re[z],Im[z]}]}
                                  ];
                          k=1;While[k<=n,{w=f[w];
                             AppendTo[listI,
                                    {RGBColor[k/n,0,0],
                                    PointSize[.08-0.07k/n],
                                    Point[{Re[w],Im[w]}]}];
                                    k++}
                                   ];
                             Return[Graphics[listI,
                                    Axes->True,
                                   AspectRatio->1
                                          ]
                                   ]
                       ]
```

2. Programme codes showing the Julia set of a rational function:

This programme code is written according to the algorithm known as the "Backward Iterate Method" which based on the theorem discussed in chapter 5. The programme codes designed for the Julia set of some selected form of rational functions in degree 2 and 3 (for those in the form $z^2 + c$ and $z^3 + c$, where c is a constant). The constant (c) and the initial point (z) can be chosen arbitrarily. The programme codes are shown as follows:

```
BackOrbit2[c_, z_]:=Block[{k,w,li,or,r,theta},
                           w=z;li={z};k=1;
                           While[k<=8000,{r=Sqrt[Abs[w-c]];
                theta=Arg[w-c]/2+Random[Integer]Pi;
                w=rCos[theta]+rSin[theta]I;
                AppendTo[li,w];k++}];
          ListPlot[Transpose[{Re[li],Im[li]}]]

BackOrbit3[c_, z_]:=Block[{k,w,li,or,r,theta},
          w=z;li={z};k=1;
          While[k<=8000,{r= Abs[w-c]^(1/3);
                theta=Arg[w-c]/3+Random[Integer,{0,2}]Pi;
                w=rCos[theta]+rSin[theta]I;
                AppendTo[li,w];k++}];
          ListPlot[Transpose[{Re[li],Im[li]}]]
```

3. Programme codes showing the Mandelbrot set.

The programme codes were written according to the algorithm based on the theorem discussed in the final section of this paper. It is quite time consuming in producing graphics by iterating each points for 1000 times, so the number of iteration had been released, as a result, the graphics so produced were a very rough approximation of the set. The number of iteration (m) and the number of points (n) inside the grid can be arbitrarily chosen. The programme code is shown as follows:

```
Mandelbrot[m_,n_]:=Block[{k,x,listI,i,j},
          listI=Table[((4i-2n)/n)+((4j-2n)/n)I,{i,0,n},{j,0,n}];
          listI=Flatten[listI];less2[x_]:=Abs[N[x]]<2;
          listI=Select[listI,less2];
          k=1;While[k<=m,{f[c_]:=Function[z,z^2+c];
                less4[c_]:=Abs[N[Nest[f[c],0,k^2]]]<4,
                listI=Select[listI,less4];k++}];
          pt[x_]:=Point[{Re[x],Im[x]}];
          listI=Map[pt,listI];
          Show[Graphics[listI,AspectRatio->1]]]
```

4. **Programme codes showing Filled Julia set of a polynomial.**

The programme codes were written according to the algorithm that points in the unbounded Fatou component will converge to ∞ upon iteration of the polynomial and we can plot a filled Julia set (the FJ). The rate of converging to ∞ can be shown by plotting a level set (The ColorFJ). The polynomial (`f`) and the number of points (`n`) inside the grid can be arbitrarily chosen. The programme are shown as follows:

```
FJ[f_,n_]:=Block[{i,j,listI,listA,k,x,y},
      listI=Table[N[((4i-2n)/n)+((4j-2n)/n)I],{i,0,n},{j,0,n}];
      listI=Flatten[listI];
      L2[x_]:=Abs[x]<2;
      listI=Select[listI,L2];
      listA={listI,listI};k=1;
            While[k<=5;{F[x_]:=N[Nest[f,x,k+1]];
                  listA[[2]]=Map[F,listA[[2]]];
                  less2[{x_,y_}]:=Abs[y]<2;
                  listA=Transpose[listA];
                  listA=Select[listA,less2];
                  listA=Transpose[listA];k++}];
            Pt[x_]:=Point[{Re[x],Im[x]}];
            Map[Pt,listA[[1]]];
      Show[Graphics[listA[[1]]]]]

ColorFJ[f_,n_]:=Block[{i,j,listI,listA,listB,k,x,y,temp},
      listI=Table[N[((4i-2n)/n)+((4j-2n)/n)I],{i,0,n},{j,0,n}];
      listI=Flatten[listI];Less2[x_]:=Abs[x]<2;
      listI=Select[listI,Less2];
      listA={listI,listI};temp={};k=1;
            While[k<=5,{F[x_]:=N[Nest[f,x,k+1]];
                  listA[[2]]=Map[F,listA[[2]]];
                  Greater2[{x_,y_}]:=Abs[y]>2;
                  listA=Transpose[listA];
                  Pt[x_]:=Point[{Re[x],Im[x]}];
                  listB=Select[listA,Greater2];
                  listA=Complement[listA,listB];
                  listB=Transpose[listB];
                  listB[[1]]=Map[Pt,listB[[1]]];
                  PrependTo[listB[[1]],CMYKColor[1,0,0,k^2/50]];
                  temp=AppendTo[temp,listB[[1]]];
                  listA=Transpose[listA];k++}];
            listA[[1]]=Map[Pt,listA[[1]]];
            temp=AppendTo[temp,listA[[1]]];
            Show[Graphics[temp]]]
```

5. Programme codes showing the iteration of sets

This programme had been adopted from [9] with some minor modifications and is used to illustrate the iteration of a chosen set, in many ocassions, it is a circle. The programme codes are shown as follows:

```
BeginPackage["ComplexMapPlot`"];

Needs["Graphics`Colors`"];

MapGraphics;
RectangularGrid;
HorizontalLines;
VerticalLines;
PolarGrid;
PolarCircles;
PolarLines;
ComplexMapPlot;

PlotPoints;
IncludePreimage;

Begin["ComplexMapPlot`Private`"];

`pairToComplex;
`complexToPair;
`transformList;
`listParameter;
`transformCircle;
`MG0;
`parametricPlotOptions;

pairToComplex[{re_, im_}] = re + im I;

complexToPair[z_] = {Re[z], Im[z]};

transformList[f_, {min:{_, _}, max:{_, _}}] :=
      Module[
            {t, minz, deltaz},
            minz = pairToComplex[min];
            deltaz = pairToComplex[max] - minz;
            ParametricPlot[
                  complexToPair[f[minz + t * deltaz]],
                  {t, 0.0, 1.0},
                  Release[parametricPlotOptions]
            ][[1, 1, 1, 1]]
      ]

transformList[f_, list_] :=
      Module[
            {t},
            ParametricPlot[
```

```
                    complexToPair[f[pairToComplex[
                        listParameter[
                            t,
                            list
                        ]
                    ]]],
                    {t, 0.0, 1.0},
                    Release[parametricPlotOptions]
                ][[1, 1, 1, 1]]
        ]

listParameter[t_, {min_, max_}] := min + t * (max - min);

listParameter[t_, list_] := Module[
        {parameter, length, startIndex, startPoint},
        length = Length[list];
        parameter = t (length - 1);
        startIndex = Floor[parameter] + 1;
        startPoint = list[[startIndex]];
        If[startIndex == length,
            startPoint,
        (*else*)
            startPoint +
                (parameter - startIndex + 1) *
                    (list[[startIndex + 1]] - startPoint)
        ]
]

transformCircle[f_, center_, {rx_, ry_}, {mint_, maxt_}] :=
        Module[
            {t, zcenter},
            zcenter = pairToComplex[center];
            ParametricPlot[
                complexToPair[f[
                    rx Cos[t] + I ry Sin[t] + zcenter
                ]],
                {t, N[mint], N[maxt]},
                Release[parametricPlotOptions]
            ][[1, 1, 1, 1]]
        ]

MapGraphics[Graphics[list_List, opts___], f_, options___] :=
        ( parametricPlotOptions =
            Join[{DisplayFunction -> Identity}, options];
        Graphics[ MG0[list, f], opts ]
        )

MG0[d_List, f_] := Map[ MG0[#, f]& , d ]

MG0[Point[d_List], f_] :=
        Point[complexToPair[f[pairToComplex[d]]]]

MG0[Line[d_List], f_] := Line[transformList[f, d]]
```

```
MG0[Rectangle[{xmin_, ymin_}, {xmax_, ymax_}], f_] :=
      MG0[Polygon[{
            {xmin,ymin},
            {xmin,ymax},
            {xmax, ymax},
            {xmax, ymin}
      }], f]

MG0[Polygon[d_List]] :=
      MG0[Polygon[Join[d, First[d]]]] /; First[d] =!= Last[d]

MG0[Polygon[d_List], f_] := Polygon[transformList[f, d]]

MG0[Circle[center_List, r_?NumberQ], f_] :=
      Line[transformCircle[
                  f, center, {r, r}, {0, 2Pi}]]

MG0[Circle[center_List, r_?List], f_] :=
      Line[transformCircle[f, center, r, {0, 2Pi}]]

MG0[Circle[center_List, r_?NumberQ, range_List], f_] :=
      Line[transformCircle[f, center, {r, r}, range]]

MG0[Circle[center_List, r_?List, range_List], f_] :=
      Line[transformCircle[f, center, r, range]]

MG0[Disk[center_List, r_?NumberQ], f_] :=
      Polygon[transformCircle[ f, center, {r, r}, {0, 2Pi}]]
MG0[Disk[center_List, r_?List], f_] :=
      Polygon[transformCircle[f, center, r, {0, 2Pi}]]

MG0[Disk[center_List, r_?NumberQ, range_List], f_] :=
      Polygon[transformCircle[ f, center, {r, r}, range]]

MG0[Disk[center_List, r_?List, range_List], f_] :=
      Polygon[transformCircle[f, center, r, range]]

MG0[RasterArray[array_, range_List:{{0,0}, {1,1}}, zrange___], f_]
:=
      RasterArray[
            array,
            {
                  complexToPair[
                        f[pairToComplex[First[range]]]],
                  complexToPair[
                        f[pairToComplex[Last[range]]]]
            },
            zrange
      ]

MG0[Text[expr_, d_List, opts___], f_] :=
      Text[expr, complexToPair[f[pairToComplex[d]]], opts]

MG0[expr_, f_] := expr
```

```
Options[RectangularGrid] =
      {
            PlotPoints -> 14
      };

RectangularGrid[{{Remin_, Remax_}, {Immin_, Immax_}},
                          options___] := Module[
      {plotPoints, n, x},

      (* Extract options *)
      {plotPoints} = {PlotPoints} /.
                  {options} /. Options[RectangularGrid];

      If[Head[plotPoints] =!= List,
            plotPoints = {plotPoints, plotPoints}
      ];

      {
      VerticalLines[{{Remin, Remax}, {Immin, Immax}},
                  PlotPoints -> plotPoints[[1]], options],
      HorizontalLines[{{Remin, Remax}, {Immin, Immax}},
                  PlotPoints -> plotPoints[[1]], options]
      }
]

Options[VerticalLines] =
      {
            PlotPoints -> 14
      };

VerticalLines[{{Remin_, Remax_}, {Immin_, Immax_}},
                          options___] := Module[
      {plotPoints, n, x},

      (* Extract options *)
      {plotPoints} = {PlotPoints} /.
                  {options} /. Options[VerticalLines];

      Flatten[Table[
            x = Remin + (Remax - Remin) (n-1)/(plotPoints-1);
            {RGBColor[1-(n-1)/(plotPoints-1),0,(n-1)/(plotPoints-
1)],

            Line[N[{{x, Immin}, {x, Immax}}]]]},
            {n, 1, plotPoints}
      ]]
]

Options[HorizontalLines] =
      {
            PlotPoints -> 14
      };

HorizontalLines[{{Remin_, Remax_}, {Immin_, Immax_}},
```

```
                                options___] :=
Module[ {plotPoints, n, x},
      (* Extract options *)
      {plotPoints} = {PlotPoints} /.
                  {options} /. Options[HorizontalLines];
      Flatten[Table[
            x = Immin + (Immax - Immin) (n-1)/(plotPoints-1);
            {CMYKColor[0,0,1,(n-1)/(2 plotPoints)+1/4],
            Line[N[{{Remin, x}, {Remax, x}}]]},
            {n, 1, plotPoints}
      ]]
]

Options[PolarGrid] =
      {
            PlotPoints -> 14
      };

PolarGrid[center:{_, _}, radius_, options___] :=
Module[ { plotPoints, ncenter, nradius, n, t,
      },

      (* Extract options *)
      {plotPoints} = {PlotPoints}
                  /. {options} /. Options[PolarGrid];

      If[Head[plotPoints] =!= List,
            plotPoints = {plotPoints, plotPoints}
      ];

      {
            GrayLevel[0.5],
            PolarLines[center, radius,
                  PlotPoints -> plotPoints[[2]], options],
            PolarCircles[center, radius,
            PlotPoints -> plotPoints[[1]], options]
      }
]

Options[PolarCircles] =
      {
            PlotPoints -> 14
      };

PolarCircles[center:{_, _}, radius_, options___] :=
Module[
      {
            plotPoints,
            nradius, ncenter, n, t,
            minradius, deltaradius
      },

      (* Extract options *)
      {plotPoints} = {PlotPoints}
```

```
                    /. {options} /. Options[PolarCircles];
        ncenter = N[center];
        nradius = N[radius];
        If[Head[nradius] =!= List, nradius = {0, nradius}];
        minradius = N[nradius[[1]]];
        deltaradius = N[nradius[[2]] - minradius];
        Flatten[Table[
              {RGBColor[(n-1)/(plotPoints-1),0,1-(n-1)/(plotPoints-
1)],
              Circle[ncenter, minradius +
                          deltaradius (n-1)/(plotPoints-1)]},
              {n, 1, plotPoints}
        ]]
]

Options[PolarLines] =
        {
              PlotPoints -> 14
        };

PolarLines[center:{_, _}, radius_, options___] :=
Module[
        {
              plotPoints,
              nradius, ncenter, n, t,
              minradius, maxradius
        },
        (* Extract options *)
        {plotPoints} = {PlotPoints}
                    /. {options} /. Options[PolarLines];
        ncenter = N[center];
        nradius = N[radius];
        If[Head[nradius]=!=List, nradius = {0, nradius}];
        minradius = N[nradius[[1]]];
        maxradius = N[nradius[[2]]];
        Flatten[Table[
              t = 2Pi n / plotPoints;
              {CMYKColor[0,0,1,(n-1)/(2 plotPoints)+1/4],
              Line[{
                    ncenter + minradius N[{Cos[t], Sin[t]}],
                    ncenter + maxradius N[{Cos[t], Sin[t]}]
              }]},
              {n, 1, plotPoints}
        ]]
]

Clear[ComplexMapPlot];

Options[ComplexMapPlot] =
        {
              AspectRatio -> Automatic,
              Axes -> Automatic,
              IncludePreimage -> True,
              Prolog -> {},
```

```
            Epilog -> {}
        };

    ComplexMapPlot[f_, var_, Graphics[domain_, ___], options___] :=
        ComplexMapPlot[f, var, domain, options];
    ComplexMapPlot[f_, var_, domain_, options___] :=
    Module[
        {output, mapOptions, showOptions,
         includePreimage},
        {includePreimage} = {IncludePreimage} /.
                            {options} /.
                            Options[ComplexMapPlot];
        mapOptions = Select[
            Join[{options}, Options[ComplexMapPlot]],
            (
                    !FreeQ[#, PlotPoints]
                 || !FreeQ[#, PlotDivision]
                 || !FreeQ[#, MaxBend]
            )&
        ];
        showOptions = Select[
            Join[{options}, Options[ComplexMapPlot]],
            (
                    FreeQ[#, PlotPoints]
                 && FreeQ[#, PlotDivision]
                 && FreeQ[#, MaxBend]
                 && FreeQ[#, Prolog]
                 && FreeQ[#, Epilog]
                 && FreeQ[#, IncludePreimage]
            )&
        ];
        output = Map[First,
            Map[
                Function[ff, MapGraphics[
                    Graphics[{domain}],
                    Function[var2, ff /. var -> var2],
                    mapOptions
                ]],
                Flatten[{f}]
            ]];
        {prolog, epilog} = {Prolog, Epilog} /. {options} /.
Options[ComplexMapPlot];
        If[TrueQ[includePreimage],
            Show[
                Graphics[{
                    Rectangle[{0,   0}, {1,   1},
                        Graphics[{domain}, showOptions]],
                    Rectangle[{1.1, 0}, {2.1, 1},
                        Graphics[{prolog, output, epilog},
                                showOptions]]
                }],
                Axes -> None,
                PlotRange -> {{0, 2.1}, {0, 1}},
                showOptions
```

```
                ],
                Show[
                     Graphics[{output, epilog}],
                     showOptions
                ]
           ]
     ]

     End[];
     EndPackage[];
```

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